Some topics on non Noetherian commutative rings

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Introduction

In Noetherian ring theory, rings characterized by homological conditions are well studied.

- Regular local rings .... $\dim A = \text{gl.dim}\, A$
- CM(Cohen–Macaulay) local rings $\dim A = \operatorname{depth} A$
- Gorenstein local rings .... $\dim A = \text{inj.dim}\, A$.

The study of the relationship between the ideals of rings and homological properties is a trend in commutative ring theory.

The homological conjecture is a collection of homological problems about finitely generated modules over Noetherian local rings.

Among them, the big CM conjecture has been studied as an important problem for many years.

Theorem 1.1 (big CM conjecture, André, 2018)

$(A, \mathfrak{m})$: *Noetherain local ring.*

There exists a A*-algebra* B *such that;*

- $B \neq \mathfrak{m}B$,
- *every system of parameters for* A *is a regular sequence on* B.

Such a B is called a big CM algebra (of A).

The big CM algebra is almost always non Noetherian if A itself is not a CM!

The André's proof uses the perfectoid algebra and the almost ring theory.

$\rightsquigarrow$ Non Noetherian rings are used as an essential tool!

In the next section, we introduce one of the main points of interest when studying non-Noetherian rings: Coherence.

Some class of non-Noetherian rings

In homological algebra, being **finitely represented** is more important than being finitely generated.

Definition 2.1

A : ring. $M \in \mathrm{Mod}(A)$ is a **finitely presented** A-module if there exist integers $n, m \in \mathbb{N}$ and an exact sequence

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0.$$

Here are some examples that demonstrate the effectiveness of being finitely presented.

Let A be a ring.

- B: flat A-algebra. For any $M, N \in \mathrm{Mod}(A)$,
$$\mathrm{Hom}_A(M, N) \otimes_A B \cong \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$
- A flat and finitely presented module is projective.
- M is a finitely presented module if and only if for any family of A-module $\{M_\lambda\}_{\lambda \in \Lambda}$, there is the natural isomorphism
$$M \otimes (\prod M_\lambda) \cong \prod (M \otimes M_\lambda).$$

Definition 2.2

A:ring. $M \in \mathrm{Mod}(A)$ is **coherent** if M is finitely generated and every finitely generated A-submodule of M is finitely presented.

A is a coherent ring if A is a coherent A-module.

Obviously, a Noetherian ring is coherent.

When A is Noetherian, the following relations hold for an A-module;

$$\text{finitely generated} \Longleftrightarrow \text{finitely presented} \Longrightarrow \text{coherent}.$$

If A is coherent, the weaker relation follows.

That is, $M \in \mathrm{Mod}(A)$ is finitely presented if and only if it is coherent.

Let me introduce the Prüfer domain as an example of a coherent ring.

Definition 2.3

A:ring. A is a **Prüfer domain** if A is a domain and it satisfies the following equivalence conditions;

(i) For any $P \in \operatorname{Spec} A$, A_P is a valuation ring.

(ii) Any finitely generated A-ideal is projective.

(iii) Any ideal is flat.

A Prüfer domain is coherent. And it is a generalization of a Dedekind domain.

$$\text{Dedekind domain} \Longleftrightarrow \text{Noetherian Prüfer domain}$$

We introduce ”torsion free”, which is weak flatness.

Definition 2.4

A : ring. $M \in \mathrm{Mod}(A)$ is a **torsion free** if for any non zero divisor of A is M-regular.

Obviously, flat modules are torsion free. So what about the opposite?

Proposition 2.5

A : domain. A is a Prüfer if and only if for any torsion free A-module is flat.

A natural generalization is the following conjecture.

Conjecture 2.6

Let A be a ring. The following two are equivalent.

(i) For any $P \in \operatorname{Spec} A$, A_P is a valuation ring.

(ii) For any torsion free A-module is flat.

In the next section, we introduce a weakly proregular sequence, that plays an important role when using local cohomology in the non Noetherian case.

Weakly proregular sequences and Schenzel's theorem

Definition 3.1

A : ring , I : ideal of A.

$H_I^i(-)$: the right derived functor of $\varinjlim \mathrm{Hom}_A(A/I^n, -)$ is called a **local cohomology**.

Note that there are following isomorphisms,

$$H_I^i(M) \cong \varinjlim \mathrm{Ext}^i(A/I^n, M)$$

since taking the inductive limit is an exact functor.

Definition 3.2

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$.

$\{e_i\}$: the standard basis of A^r.

For each $I = \{j_1, \ldots, j_i\}$ $(1 \leq j_1 < \cdots < j_i \leq r)$, let $a_I = a_{j_1} \cdots a_{j_i}$ and $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$.

$C^\bullet(\underline{a})$: the complex defined by;

$$C^i(\underline{a}) := \sum_{\#I=i} A_{a_I} e_I,$$

$$d^i : C^i(\underline{a}) \to C^{i+1}(\underline{a}); e_I \mapsto \sum_{j=1}^{r} e_I \wedge e_j.$$

It is called a **Čech complex**.

$\check{H}^i(\underline{a})$: the cohomology of $C^\bullet(\underline{a})$ is called a **Čech cohomology**.

For $M \in \mathrm{Mod}(A)$, we define $C^\bullet(\underline{a}, M) := C^\bullet(\underline{a}) \otimes M, \check{H}^i(\underline{a}, M) := H^i(C^\bullet(\underline{a}, M))$.

The relationship between Čech cohomology and Koszul cohomology is important.

For $\underline{a} = a_1, \ldots, a_r \in A$, we let $\underline{a}^n := a_1^n, \ldots, a_r^n$, then the following isomorphisms hold for any $M \in \mathrm{Mod}(A)$;

$$\check{H}^i(\underline{a}, M) \cong \varinjlim_n H^i(\underline{a}^n, M).$$

These Čech and Koszul cohomologies are somewhat easy to compute.

Theorem 3.3

A : Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. There are isomorphisms;

$$H^i_I(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$.

What happens if we remove the Noetherian assumption? Can we extend this theorem?

$\rightsquigarrow$ This theorem was extended by Schenzel (2003) by introducing a weakly proregular sequence.

A weakly proregular sequence is defined by using Koszul homologies.

Definition 3.4 (Schenzel)

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$.

$\underline{a}$ is called a **weakly proregular sequence** if
$1 \leq {}^\forall i \leq r, {}^\forall n \geq 0, {}^\exists m \geq n; \varphi_{mn} : H_i(\underline{a}^m) \to H_i(\underline{a}^n)$ is the zero map.

Theorem 3.5 (Schenzel)

A : ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$.

$$\underline{a} \text{ is a weakly proregular sequence} \iff {}^\forall i \geq 0, {}^\forall M \in \mathrm{Mod}(A), H^i_I(M) \cong \check{H}^i(\underline{a}, M) .$$

We will explain that Schenzel's theorem is an extension of the Noetherian case.

Proposition 3.6

A: Noetherian ring , $\underline{a} = a_1, \ldots, a_r \in A$. $\underline{a}$ is a weakly proregular sequence.

Corollary 3.7

A : Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. There are functorial isomorphisms;

$$H^i_I(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$ and $i \geq 0$.

Schenzel proved his theorem (Theorem 3.5) using the derived category theory.

In Ando (2022), we gave a simpler proof using the Abelian category theory, without using the derived one.

The key is to prove the following proposition in the Abelian category theory.

Proposition 3.8 (A.)

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$.

$$\underline{a} \text{ is a weakly proregular seqence} \Longleftrightarrow \check{H}^{\bullet}(\underline{a}, -) \text{ is an effaceable } \delta\text{-functor.}$$

This proposition can be proved using only the Abelian category theory.

Recently, weakly proregular sequences have been apllied to perfectoid algebras,
in paticular to the Noetherian ring theory of mixied characteristic, by Bhatt, Iyengar and Ma (2019).

Also, it is proposed to generalize CM rings to non-Noetherian rings by using weakly proregular sequences by Hamilton and Marley (2007).

There is still much we do not know about Hamilton–Marley's definition of CM rings.

Conjecture 3.9

Let A be a CM ring in the sense of Hamilton–Marley. $A[X]$ is also CM.

The following is known about this conjecture.

Proposition 3.10 (Kim and Walker (2020))

Let A be a finite dimensional Prüfer domain. $A[X_1, \ldots, X_n]$ is CM for $n \geq 1$.

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