Weakly proregular sequences and their applications, regularity criteria, and the Cohen-Macaulay property

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Throughout this talk, a ring is assumed to be a commutative ring with 1, not necessarily Noetherian. We also let $p > 0$ be a prime number.

In Noetherian ring theory, rings characterized by homological conditions are well-studied.

- Regular local rings .... $\dim A = \mathrm{gl.dim}\, A$
- CM (Cohen–Macaulay) local rings $\dim A = \operatorname{depth} A$
- Gorenstein local rings .... $\dim A = \mathrm{inj.dim}\, A$.

The study of the relationship between the ideals of rings and homological properties is a major trend in commutative ring theory.

- The homological conjectures are a family of problems concerning finitely generated modules over Noetherian local rings.
- Among them, the Big Cohen–Macaulay Conjecture has long been regarded as a central problem.

Theorem 1.1 (Big CM conjecture, André, 2018)

Let $(A, \mathfrak{m})$ be a Noetherian local ring.
There exists an A-algebra B such that:

- $B \neq \mathfrak{m}B$,
- *every system of parameters for A is a regular sequence on B.*

Such a B is called a big CM algebra of A.

- The homological conjectures were resolved classically in the equicharacteristic case.
- However, the mixed characteristic case is more difficult.
- André resolved the big CM conjecture by making full use of perfectoid algebras.

Recall 1.2

$(A, \mathfrak{m}, k)$: local ring. The characteristic $\mathrm{char}\, A$ of a ring A is restricted to the following four cases.

	$\mathrm{char}\, A$	$\mathrm{char}\, k$
equichar. 0	0	0
equichar. p	p	p
mixed char. $(0, p)$	0	p
mixed char. $(p^n, p), (n > 1)$	p^n	p

Table: Pairings of rings and their characteristic

- In the equicharacteristic case, the Frobenius map $F : A \longrightarrow A;\ a \mapsto a^p$ is a ring homomorphism, and the analysis of this map is a critical technique. A is **perfect** if the Frobenius map $F : A \longrightarrow A$ is bijective.
- Extending this concept to the mixed characteristic case leads to the notion of **perfectoid**.

Example 1

A standard example of a perfectoid ring constructed from p-adic integers is $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$.
This ring is obtained by adjoining all p-power roots of p to $\mathbb{Z}_p$ and taking the p-adic completion.

- A perfectoid ring is a class of well-behaved algebras over a ring A.
- However, since such a ring is no longer Noetherian, it is necessary to develop a theory that works even for non-Noetherian rings.
- For example, there exists a class of rings called coherent rings that generalizes Noetherian rings.

Kunz's theorem, which is powerful in the equicharacteristic case, can be extended to the mixed characteristic case by means of perfectoid techniques.

Theorem 1.3 ([Kun76])

A: Noetherian ring with $\operatorname{char} A = p > 0$.
The following conditions are equivalent.

1. *A is regular.*
2. *$F : A \longrightarrow A; a \mapsto a^p$ is flat.*

Theorem 1.4 ([BIM19,Theorem 4.7.])

A: Noetherian ring with $p \in \operatorname{rad} A$.
The following conditions are equivalent.

1. *A is regular.*
2. *There exists a faithfully flat ring homomorphism $A \longrightarrow B$ with B perfectoid.*

In the next section, we introduce **weakly proregular sequences**.

- Applied in [BIM19] to investigate the vanishing of Tor over A_{perf} and A^+, which is necessary to determine regularity.
- [HM07] attempts to generalize the Cohen–Macaulay property to non-Noetherian rings. In this context, weakly proregular sequences are used to define a generalization of systems of parameters.

Definition 2.1

A: ring, I: ideal of A.
$H_I^i(-)$: the right derived functor of $\varinjlim \mathrm{Hom}_A(A/I^n, -)$ is called **local cohomology**.

Note that there are the following isomorphisms,

$$H_I^i(M) \cong \varinjlim \mathrm{Ext}^i(A/I^n, M)$$

since taking the inductive limit is an exact functor.

Definition 2.2

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$.
$\{e_i\}$: the standard basis of A^r.
For each $I = \{j_1, \ldots, j_i\}$ $(1 \leq j_1 < \cdots < j_i \leq r)$, let $a_I = a_{j_1} \cdots a_{j_i}$ and $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$.
$C^\bullet(\underline{a})$: the complex defined by:

$$C^i(\underline{a}) := \sum_{\#I=i} A_{a_I} e_I,$$

$$d^i : C^i(\underline{a}) \to C^{i+1}(\underline{a}); e_I \mapsto \sum_{j=1}^{r} e_I \wedge e_j.$$

It is called the **Čech complex**.
$\check{H}^i(\underline{a})$: the cohomology of $C^\bullet(\underline{a})$ is called **Čech cohomology**.

For $M \in \mathrm{Mod}(A)$, we define $C^\bullet(\underline{a}, M) := C^\bullet(\underline{a}) \otimes M$, $\check{H}^i(\underline{a}, M) := H^i(C^\bullet(\underline{a}, M))$.

Theorem 2.3

A: Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. There are isomorphisms

$$H^i_I(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$.

What happens if we remove the **Noetherian assumption**?

$\rightsquigarrow$ Schenzel [Sch03] extended the theorem by introducing a **weakly proregular sequence**.

A weakly proregular sequence is defined using Koszul homology.

Definition 2.4 (Schenzel)

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$.
$\underline{a}$ is called a **weakly proregular sequence** if
$1 \leq {}^\forall i \leq r, {}^\forall n \geq 0, {}^\exists m \geq n; \varphi_{mn} : H_i(\underline{a}^m) \to H_i(\underline{a}^n)$ is the zero map.

Theorem 2.5 (Schenzel)

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$.

$$\underline{a} \text{ is a weakly proregular sequence} \iff {}^\forall i \geq 0, {}^\forall M \in \mathrm{Mod}(A), H^i_I(M) \cong \check{H}^i(\underline{a}, M).$$

We will explain that Schenzel's theorem is an extension of the Noetherian case.

Proposition 2.6

A: Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$. $\underline{a}$ is a weakly proregular sequence.

Corollary 2.7

A: Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$.
There are functorial isomorphisms

$$H_I^i(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$ and $i \geq 0$.

- Schenzel proved his theorem (Theorem 2.5) using derived category theory.
- In [And22], we gave a simpler proof using Abelian category theory, without relying on derived category theory.

$\rightsquigarrow$ This result ensures that set-theoretic subtleties, such as Grothendieck universes, can be disregarded when computing local cohomology groups using Schenzel's theorem.

★ The key is establishing the following proposition within Abelian category theory.

Proposition 2.8 (A.)

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$.

$$\underline{a} \text{ is a weakly proregular seqence} \iff \text{For } i > 0,\ \check{H}^i(\underline{a}, -) \text{ is an effaceable functor.}$$

This proposition can be proved using only Abelian category theory.

- This proposition can be proven by appropriately selecting injective modules for each i and computing the Koszul homology.
- By this proposition and Grothendieck's theorem, $\check{H}^\bullet(\underline{a}, -)$ is a universal δ-functor.
- Since $H^0_I(-) = \check{H}^0(\underline{a}, -)$ holds by definition, the uniqueness of universal δ-functors implies that $H^i_I(-) = R^i(H^0_I(-)) = \check{H}^i(\underline{a}, -)$ for all $i > 0$.

Recently, weakly proregular sequences have been applied to non-Noetherian commutative ring theory.

- In [BIM19], they are used to characterize regularity.

Theorem 2.9 ([BIM19, Theorem 4.13])

Let $(A, \mathfrak{m}, k)$ be an excellent local domain. Then A is regular if any of the following conditions hold:

1. *A has positive characteristic and $\mathrm{Tor}_i^A(A_{\mathrm{perf}}, k) = 0$ for some $i \geq 1$.*
2. *A has positive characteristic and $\mathrm{Tor}_i^A(A^+, k) = 0$ for some $i \geq 1$.*
3. *A has mixed characteristic, $\dim A \leq 3$, and $\mathrm{Tor}_i^A(A^+, k) = 0$ for some $i \geq 1$.*

Let $\underline{a}$ be a system of parameters of A.

- If A has positive characteristic, then $\underline{a}$ is weakly proregular on A_{perf} and A^+.
- If A has mixed characteristic and $\dim A \leq 3$, then $\underline{a}$ is weakly proregular on A^+.

- In Noetherian ring theory, Cohen–Macaulay rings have been actively studied as a particularly well-behaved class of rings.
- For a Noetherian local ring A, the following conditions are equivalent.

A ring satisfying these conditions is called a **Cohen–Macaulay ring**.

1. $\dim A = \operatorname{depth} A$, where $\operatorname{depth} A$ is the maximal length of a regular sequence contained in the maximal ideal of A.
2. Every system of parameters is a regular sequence.
3. For every (proper) ideal I, $\operatorname{ht} I = \operatorname{grade} I$.

- A naive generalization of these concepts does not work well. For example, consider a valuation ring.
    - For any valuation ring V (that is not a field), $\operatorname{depth} V = 1$.
    - For a non-Noetherian valuation ring V, $\dim V \geq 2$.
- However, every valuation ring is a coherent regular ring. This notion is a generalization of regular rings, and implies the Cohen–Macaulay property in the Noetherian setting. Thus, we expect valuation rings to be Cohen–Macaulay.
- Therefore, it is not natural to generalize the Cohen–Macaulay property to non-Noetherian rings based on the condition $\dim A = \operatorname{depth} A$.

Therefore, we seek a generalization to non-Noetherian rings satisfying the following conditions:

1. The definition agrees in the Noetherian case.
2. Regular coherent rings are Cohen–Macaulay.
3. A is Cohen–Macaulay iff $A[X]$ is Cohen–Macaulay.
4. A is Cohen–Macaulay iff A_P is Cohen–Macaulay for all $P \in \operatorname{Spec} A$.

In [HM07], a definition satisfying conditions 1 and 2, as well as the "if" parts of 3 and 4, was proposed by generalizing systems of parameters using weakly proregular sequences. (The "only if" parts remain unknown.)

Definition 3.1 ([HM07,Definition 3.1, 4.1])

Let A be a ring and M an A-module. A finite sequence $\underline{a} = a_1, \ldots, a_r \in A$ is called a **parameter sequence** if the following conditions hold:

1. $\underline{a}$ is weakly proregular;
2. $\underline{a}A \neq A$;
3. $\check{H}^r(\underline{a}, A)_P \neq 0$ for all prime ideals P containing $\underline{a}$.

The sequence $\underline{a}$ is called a **strong parameter sequence** if $a_1, \ldots, a_i$ is a parameter sequence for $i = 1, \ldots, r$. A ring A is called **Cohen–Macaulay** if every strong parameter sequence is a regular sequence.

- If A is Noetherian, the notions of a strong parameter sequence and a system of parameters coincide.
- In particular, this definition is a generalization of the definition for Noetherian rings.

- [HM07] proves that several classes of rings are indeed Cohen–Macaulay.For example: A^+, where A is an excellent Noetherian domain of characteristic $p > 0$.
- Their proofs utilize **polynomial grade**, an extension of classical grade introduced by Hochster [Hoc74].
- We will present some propositions related to this concept, as well as counterexamples to certain claims made in [HM07].

Definition 4.1

Let A be a ring, and M an A-module. Let $\underline{a} = a_1, \ldots, a_r \in A$. The sequence $\underline{a}$ is called a **weak M-sequence** if a_i is a non-zerodivisor on $M/(a_1, \ldots, a_{i-1})M$ for each i.

- If $M/\underline{a}M \neq 0$, a weak M-sequence is simply a regular sequence.
- We follow the terminology of Bruns and Herzog (1997).
- [Hoc74] refers to this as a "possibly improper regular sequence on M."

Let $\mathrm{grade}_I(M)$ denote the maximal length of a weak M-sequence contained in an ideal I.

Lemma 4.2

Let A be a Noetherian ring and M a finitely generated A-module.

$$\mathrm{grade}_I(M) > 0 \iff (0 :_M I) := \{x \in M \mid Ix = 0\} = 0$$

- However, without the Noetherian assumption, it is possible that $\mathrm{grade}_I(M) = 0$ even if $(0 :_M I) = 0$.
- That is, for every $a \in I$, there exists a non-zero $x \in M$ such that $ax = 0$, yet for every non-zero $x \in M$, there exists $a \in I$ such that $ax \neq 0$.
- Such examples can be constructed using the trivial extension technique due to Nagata.

Example 2 ([Vas71])

Let k be a field, $A = k[[x, y]]$, $\mathfrak{m} = (x, y)$, and $M = \bigoplus_{P \in \mathrm{Spec}\, A, \mathrm{ht}\, P = 1} A/P$. Then, in the trivial extension $A * M$, we have $(0 :_{A*M} \mathfrak{m} * M) = 0$, but $\mathrm{grade}_{\mathfrak{m}*M}(A * M) = 0$.

In fact, this property can be recovered by extending the ring to a polynomial ring.

Lemma 4.3 ([Nor76, Chap. 5, Thm. 7])

Let A be a ring, $I = (a_1, \ldots, a_r)$, and M an A-module. $\operatorname{grade}_{IA[X]}(M \otimes_A A[X]) > 0$ *if and only if* $(0 :_M I) := \{x \in M \mid Ix = 0\} = 0$.

This motivates the following definition.

Definition 4.4 ([Nor76, Chap. 5.5])

$$\operatorname{p-grade}_I M := \lim_{n \to \infty} \operatorname{grade}_{IA[X_1, \ldots, X_n]}(M[X_1, \ldots, X_n])$$

This is called the **polynomial grade** of M with respect to I.

In general, the following inequality holds:

$$\operatorname{p-grade}_I M \le \sup \left\{ \operatorname{grade}_{IB}(M \otimes_A B) \;\middle|\; B : \text{faithfully flat } A\text{-algebra} \right\}.$$

We want to show that equality holds. Since this fact is stated without proof in [HM07], we first provide a complete proof.

Definition 4.5 ([Hoc74])

The pair (I, M) is called **admissible** if, for any faithfully flat A-algebra B, every weak $M \otimes B$-sequence contained in IB is an $M \otimes B$-regular sequence.

- If $IM \neq M$, then (I, M) is admissible.
- Moreover, if M is finitely generated, then $IM \neq M$ is equivalent to (I, M) being admissible.

Proposition 4.6 ([Hoc74, Sect. 1, Prop. 2])

Let A be a ring, I an ideal, M an A-module, and B a faithfully flat A-algebra. Assume that (I, M) is admissible. Then there exists $n \geq 0$ such that

$$\mathrm{grade}_{IB}(M \otimes_A B) \leq \mathrm{grade}_{IA[X_1,\ldots,X_n]}(M[X_1, \ldots, X_n]).$$

Thus, we see that the equality $\lim \mathrm{grade}(M[X_1, \ldots, X_n]) = \sup\{\mathrm{grade}(M \otimes B)\}$ holds when (I, M) is admissible.

Proposition 4.7

Let M be finitely generated, and assume $IM = M$. Then $\operatorname{grade}_I M = \infty$. In particular, $\lim_{n\to\infty} \operatorname{grade}_{IA[X_1,\ldots,X_n]}(M[X_1,\ldots,X_n]) = \infty$, so the desired equality holds.

We constructed the following important example for the case where $IM = M$ but M is not finitely generated.

Example 3

Let $A = \mathbb{Z}$, $I = 2\mathbb{Z}$, and $M = \{a/2^n + \mathbb{Z} \mid a \in \mathbb{Z}, n \geq 0\} \subset \mathbb{Q}/\mathbb{Z}$. Then $IM = M$, yet no element of I is M-regular. That is, $\operatorname{grade}_I M = 0$. In this case, since I is principal, $\operatorname{grade}_{IB}(M \otimes B) = 0$ for any faithfully flat A-algebra B (consequently, the equality holds with both sides being 0).

In fact, when (I, M) is not admissible, the following holds:

Lemma 4.8

Let A be a ring, I an ideal, and M an A-module. If (I, M) is not admissible, then

$$\sup \left\{ \operatorname{grade}_{IB}(M \otimes_A B) \;\middle|\; B : \textit{faithfully flat } A\textit{-algebra} \right\} = \infty.$$

Therefore, it suffices to show that if (I, M) is not admissible, then $\lim \operatorname{grade}(M[X_1, \ldots, X_n]) = \infty$.

Proposition 4.9

Suppose that (I, M) is not admissible. Then $\lim_{n\to\infty} \mathrm{grade}_{IA[X_1,\dots,X_n]}(M[X_1, \dots, X_n]) = \infty$.

Proof.

By [Hoc74,Sect. 1, Prop. 3], there exists a sequence $\underline{a} = a_1, \dots, a_r \subset I$ such that $H_i(\underline{a}, M) = 0$ for all i. Consider the Koszul complex $K_\bullet(\underline{a}, M)$, where the boundary map is given by $d_r : M \to M^r; x \mapsto (a_1x, -a_2x, \dots, \pm a_r x)$. Since all Koszul homology groups vanish, we have $H_r(\underline{a}, M) = \ker d_r = \{x \in M \mid a_i x = 0 \text{ for all } i\} = 0$. Therefore, $u_1 := \sum_{j=1}^r a_j X_1^j$ is a regular element on $M[X_1]$. For each i, we have $H_i^{A[X_1]}(\underline{a}, M[X_1]) = H_i^A(\underline{a}, M) \otimes A[X_1] = 0$. Considering the long exact sequence of Koszul homology induced by the short exact sequence

$$0 \longrightarrow M[X_1] \xrightarrow{u_1 \cdot} M[X_1] \longrightarrow M[X_1]/u_1M[X_1] \longrightarrow 0$$

we obtain $H_i^{A[X_1]}(\underline{a}, M[X_1]/u_1M[X_1]) = 0$. Similarly, let $u_2 := \sum_{j=1}^r a_j X_2^j$. This is a regular element on $M[X_1]/u_1M[X_1]$, and we have $H_i^{A[X_1,X_2]}(\underline{a}, M[X_1, X_2]/(u_1, u_2)M[X_1, X_2]) = 0$. Repeating this process shows that $\lim \mathrm{grade}(M[X_1, \dots, X_n]) = \infty$. □

Consequently, we can prove the desired equality.

Theorem 4.10

Let A be a ring, I an ideal of A, and M an A-module. Then

$$\lim_{n\to\infty} \operatorname{grade}_{IA[X_1,\dots,X_n]} M[X_1,\dots,X_n] = \sup\left\{\operatorname{grade}_{IB}(M\otimes_A B) \,\middle|\, B : \textit{faithfully flat } A\textit{-algebra}\right\}.$$

Proof.

If (I, M) is admissible, the equality holds by Proposition 4.6. If (I, M) is not admissible, the equality holds by Proposition 4.9. □

Finally, I would like to mention a counterexample to a claim made in [HM07].

Proposition 4.11 ([HM07, Prop. 2.7])

Let A be a ring, $\underline{a}$ a finite sequence of elements from A of length $\ell = \ell(\underline{a})$, $I = \underline{a}A$, and M an A-module. The following integers (including the possibility of ∞) are equal:

(1) $\operatorname{p-grade}_I(M)$;

(2) $\sup\left\{k \geq 0 \,\middle|\, H_{\ell-i}(\underline{a}, M) = 0 \text{ for all } i < k\right\}$;

(3) $\sup\left\{k \geq 0 \,\middle|\, \check{H}^i_I(M) = 0 \text{ for all } i < k\right\}$.

(Note: The following statement is incorrect.) Moreover, $\operatorname{p-grade}_I(M) < \infty$ if and only if $IM \neq M$.

In particular, when I is finitely generated, this implies that $IM = M$ if and only if $\operatorname{p-grade}_I(M) = \infty$. However, this is incorrect. Example 3 serves as a counterexample. In this example, we have $\operatorname{p-grade}_I(M) = 0$. Computing the Koszul and Čech cohomologies shows that the values in (2) and (3) are also 0.

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