Cohomological Cohen–Macaulayness in Non-Noetherian Rings

Applications to big Cohen–Macaulay algebras and absolute integral closures

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2026 International workshop on Algebra at TUS-Noda

November 25, 2026

Based on arXiv:2608.17879.

Cohen–Macaulayness for Noetherian rings

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. The following conditions are equivalent:

1. A is Cohen–Macaulay, i.e., $\operatorname{depth} A = d$.
2. Every **system of parameters** of A is a regular sequence.
3. $H^i_{\mathfrak{m}}(A) = 0 \qquad (i < d)$.

We would like to extend Cohen–Macaulayness to non-Noetherian rings. However, these characterizations are no longer equivalent in general.

Two approaches: regular sequences and cohomology.

Absolute integral closures and big CM algebras

Let A be a domain and $K = \operatorname{Frac} A$. The integral closure of A in an algebraic closure of K is its **absolute integral closure**, denoted by A^+.

Hochster–Huneke [HH92]

If A is an excellent Noetherian domain of positive characteristic, then A^+ is a **balanced big Cohen–Macaulay A-algebra**.

This means that systems of parameters **of** A act as regular sequences on A^+: a property **relative to** A. But A^+ is usually non-Noetherian. How should we define Cohen–Macaulayness of the ring A^+ itself?

Why Hamilton–Marley CM?

For non-Noetherian rings, we would like to generalize the criterion **"every system of parameters is regular."**

Hamilton–Marley's approach [HM07]

Using **weakly proregular sequences** and Čech cohomology, they define **parameter sequences**.

A ring is **HMCM** if every **strong parameter sequence** is a regular sequence.

Hamilton–Marley [HM07]

If A is an excellent Noetherian domain of positive characteristic, then A^+ **is HMCM**.

From a **relative** big CM property over A to a CM property **of** A^+ **itself**.

HMCM and the localization problem

Cohen–Macaulayness of Noetherian rings is preserved under localization. Is the same true for HMCM?

$$A \text{ is HMCM} \overset{?}{\Longrightarrow} A_{\mathfrak{p}} \text{ is HMCM} \qquad (\mathfrak{p} \in \operatorname{Spec} A).$$

Counterexample (A.) [And26b]

There exist an HMCM ring A and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $A_{\mathfrak{p}}$ is not HMCM.

Thus Hamilton–Marley's result alone does not imply that A^+ is locally HMCM. **Question:** Is A^+ nevertheless locally HMCM?

Goal and strategy

Goal

For an excellent Noetherian domain A of positive characteristic, prove that A^+ is locally HMCM.

Motivated by Bhatt's cohomological Cohen–Macaulayness, we introduce a stronger property called **CCM**.

$$\begin{array}{c} A \to R \text{ is integral} \\ R \text{ is a locally balanced} \\ \text{big CM } A\text{-algebra} \end{array} \longrightarrow R \text{ is CCM} \longrightarrow \begin{array}{c} R \text{ is} \\ \text{locally HMCM} \end{array}$$

We prove both implications and apply them to A^+.

Three kinds of cohomology

Let A be a ring, $I = (a_1, \dots, a_r)$, and M an A-module.

- **Čech cohomology**: $\check{H}^i(\underline{a}, M)$.
- **Module-theoretic local cohomology**: $H^i_I(M)$, the right derived functors of Γ_I.
- **Sheaf cohomology with support**: $H^i_{V(I)}(\operatorname{Spec} A, \widetilde{M})$.

For any finite sequence $\underline{a}$,

$$H^i_{V(I)}(\operatorname{Spec} A, \widetilde{M}) \cong \check{H}^i(\underline{a}, M).$$

By contrast, identifying $H^i_I(M)$ with $\check{H}^i(\underline{a}, M)$ requires an additional hypothesis in general.

Weakly proregular sequences

Let $\underline{a} = a_1, \ldots, a_r \in A$.

Definition 1

The sequence $\underline{a}$ is **weakly proregular** if, for all $i > 0$ and $n \geq 1$, there exists $m \geq n$ such that

$$H_i(\underline{a}^m; A) \longrightarrow H_i(\underline{a}^n; A)$$

is the zero map.

By Schenzel, this is equivalent to the existence of natural isomorphisms, for every A-module M,

$$H^i_I(M) \cong \check{H}^i(\underline{a}, M) \qquad (i \geq 0).$$

Every finite sequence over a Noetherian ring is weakly proregular.

Parameter sequences

Definition 2 (Hamilton–Marley)

A finite sequence $\underline{a} = a_1, \ldots, a_r \in A$ is a **parameter sequence** if:

1. $\underline{a}$ is weakly proregular;
2. $(a_1, \ldots, a_r)A \neq A$;
3. $\check{H}^r(\underline{a}, A)_P \neq 0$ for every $P \in V(\underline{a}A)$.

It is a **strong parameter sequence** if every initial segment $a_1, \ldots, a_j$ $(1 \leq j \leq r)$ is a parameter sequence.

Hamilton–Marley Cohen–Macaulayness

Definition 3 (Hamilton–Marley)

A ring A is **HMCM** if every strong parameter sequence in A is a regular sequence.

Equivalent criterion [HM07]

A is HMCM if and only if, for every strong parameter sequence $\underline{a}$ of length r,

$$\check{H}^i(\underline{a}, A) = 0 \qquad (i < r).$$

We call A **locally HMCM** if A_P is HMCM for every $P \in \operatorname{Spec} A$. Every coherent regular ring is locally HMCM.

CCM: the cohomological approach

A ring A is **locally finite-dimensional** if $\dim A_P < \infty$ for every $P \in \operatorname{Spec} A$.

Definition 4 (Cohomological Cohen–Macaulayness)

A locally finite-dimensional ring A is **CCM** if, for every $P \in \operatorname{Spec} A$,

$$H^i_{PA_P}(\operatorname{Spec} A_P, \mathcal{O}) = 0 \qquad (i < \dim A_P).$$

The condition examines sheaf cohomology supported at the closed point of each local ring, following Bhatt's approach.

Basic properties of CCM (A.)

Proposition 5

1. *CCM is preserved under localization.*
2. *For a Noetherian ring A,*

$$A \text{ is CCM} \iff A \text{ is Cohen–Macaulay}.$$

Why?

- CCM is defined in terms of all local rings A_P.
- In the Noetherian case, sheaf cohomology with support agrees with ordinary local cohomology, which detects depth.

Thus CCM has the **stability under localization** that HMCM lacks.

Comparison theorem (A.): CCM $\Rightarrow$ locally HMCM

Theorem 6

*Every **locally finite-dimensional CCM ring** A is **locally HMCM**.*

In fact, we prove a stronger vanishing statement:

Stronger vanishing

For every $P \in \operatorname{Spec} A$ and every **parameter sequence** $\underline{a}$ of length r in A_P,

$$\check{H}^i(\underline{a}, A_P) = 0 \qquad (i < r).$$

Key idea: Choose a minimal prime in the support of nonzero Čech cohomology and compare degrees using CCM.

Proof of the comparison theorem I: minimal support

It suffices to consider a **finite-dimensional CCM local ring** A. Let $\underline{a}$ be a parameter sequence of length r.

$$i \coloneqq \min\{j \mid \check{H}^j(\underline{a}, A) \neq 0\}, \qquad N \coloneqq \check{H}^i(\underline{a}, A).$$

By finite-dimensionality, $\operatorname{Supp} N$ has a minimal element P.

Properties of P

- $N_P \neq 0$.
- The support of $\widetilde{N_P}$ on $\operatorname{Spec} A_P$ is exactly the closed point $\{PA_P\}$.
- $\check{H}^j(\underline{a}, A) = 0$ for $j < i$.

We exploit this concentration of support in a spectral sequence.

Proof of the comparison theorem II: spectral sequence

Put $X = \operatorname{Spec} A_P$ and $Z = V(\underline{a}A_P)$. We have a Grothendieck spectral sequence:

$$E_2^{p,q} = H^p_{PA_P}\big(X, \mathcal{H}^q_Z(\mathcal{O}_X)\big) \implies H^{p+q}_{PA_P}(X, \mathcal{O}_X).$$

- All rows with $q < i$ vanish.
- For $q = i$, $\mathcal{H}^i_Z(\mathcal{O}_X) \cong \widetilde{N_P}$.
- Since $\widetilde{N_P}$ is supported only at the closed point,

$$E_2^{0,i} = N_P \neq 0, \qquad E_2^{p,i} = 0 \quad (p > 0).$$

Therefore $E_2^{0,i}$ survives to E_∞, giving

$$\boxed{H^i_{PA_P}(X, \mathcal{O}_X) \cong N_P \neq 0.}$$

Proof of the comparison theorem III: degree bounds

From CCM and the preceding nonvanishing, we obtain

$$\dim A_P \leq i.$$

On the other hand, since $\underline{a}$ is a parameter sequence of length r, Hamilton–Marley's height inequality gives

$$i \leq r \leq \dim A_P.$$

Combining the inequalities,

$$\boxed{\dim A_P \leq i \leq r \leq \dim A_P \quad \Longrightarrow \quad i = r.}$$

Hence $\check{H}^j(\underline{a}, A) = 0$ for $j < r$. The HMCM criterion applies to every local ring. □

The converse fails: valuation rings (A.)

Proposition 7

If V is a finite-dimensional valuation ring, then

$$V \text{ is CCM} \iff \dim V \leq 1.$$

Yet every valuation ring is coherent regular, hence locally HMCM.

For example, suppose $\dim V = 2$, with maximal ideal $\mathfrak{m}$. For a suitable $f \in \mathfrak{m}$,

$$H^1_{\mathfrak{m}}(\operatorname{Spec} V, \mathcal{O}) \cong V_f/V \neq 0.$$

Thus V is not CCM [Dat17].

CCM is strictly stronger than locally HMCM.

Locally balanced big Cohen–Macaulay algebras

Let A be Noetherian and R an A-algebra.

Definition 8

We say R is a **locally balanced big CM A-algebra** if, for every $P \in \operatorname{Spec} A$, we have $R_P/PR_P \neq 0$ and every **system of parameters** of A_P is an R_P-regular sequence.

This is a condition on regular sequences **over** A, whereas CCM concerns the cohomology **of** R **as a ring**.

Question

If, in addition, $A \to R$ **is integral**, does it follow that

$$\boxed{R \text{ is CCM?}}$$

Application theorem (A.): integral big CM algebras

Theorem 9

*Let A be a Noetherian ring and R an **integral** A-algebra which is **locally balanced big Cohen–Macaulay** over A. Then*

$$\boxed{R \text{ is CCM, and hence locally HMCM.}}$$

Outline of the proof:

- At each $R_{\mathfrak{p}}$, use a system of parameters of A_P.
- Incomparability for integral extensions ensures that its vanishing locus consists only of the closed point.
- Regularity yields vanishing of cohomology with support.

Proof of the application theorem I: the closed point

Fix $\mathfrak{p} \in \operatorname{Spec} R$, let $P = \mathfrak{p} \cap A$, and set $d = \dim A_P$. Let $\underline{a} = a_1, \ldots, a_d$ be a system of parameters of A_P. By assumption, it is an $R_{\mathfrak{p}}$-regular sequence.

Crucial equality

$$V(\underline{a}R_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}.$$

Indeed, if $\underline{a}R_{\mathfrak{p}} \subseteq \mathfrak{q}R_{\mathfrak{p}}$, then $\sqrt{\underline{a}A_P} = PA_P$ implies $\mathfrak{q} \cap A = P$. By **incomparability** for the integral map $A \longrightarrow R$, the inclusions $\mathfrak{q} \subseteq \mathfrak{p}$ and equal contractions imply $\mathfrak{q} = \mathfrak{p}$.

Proof of the application theorem II: vanishing

Since $\underline{a}$ is an $R_{\mathfrak{p}}$-regular sequence,

$$\check{H}^i(\underline{a}, R_{\mathfrak{p}}) = 0 \qquad (i < d).$$

Using the equality of supports and the Čech description,

$$\begin{aligned} H^i_{\mathfrak{p}R_{\mathfrak{p}}}(\operatorname{Spec} R_{\mathfrak{p}}, \mathcal{O}) &\cong \check{H}^i(\underline{a}, R_{\mathfrak{p}}) \\ &= 0 \qquad (i < d). \end{aligned}$$

Since R is integral over A, $\dim R_{\mathfrak{p}} \leq d < \infty$. Thus R is CCM, and the comparison theorem implies that R is locally HMCM. □

Application I (A.): absolute integral closure in char. p

Corollary 10

*Let A be an **excellent Noetherian domain of positive characteristic**. Then A^+ is **CCM** and, in particular, **locally HMCM**.*

- Hochster–Huneke: $(A_P)^+$ is a balanced big CM A_P-algebra for every $P \in \operatorname{Spec} A$.
- Absolute integral closure commutes with localization:

$$(A^+)_P \cong (A_P)^+.$$

- Since $A \to A^+$ is integral, the application theorem applies.

This strengthens Hamilton–Marley's result "A^+ **is HMCM**" to "A^+ is locally HMCM."

Application II (A.): A^+/p^nA^+

Corollary 11

Let A be an excellent Noetherian domain, p a prime number, and $n \geq 1$. Assume $A/pA \neq 0$. Then

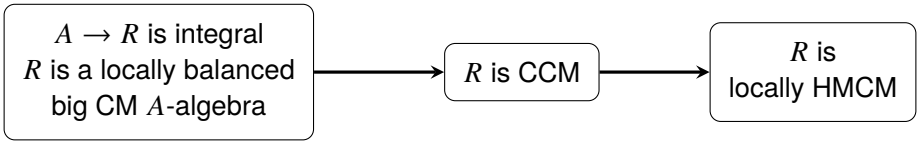
$$\boxed{A^+/p^nA^+ \text{ is CCM and locally HMCM.}}$$

- Bhatt: A^+/p^nA^+ is a locally balanced big CM algebra over A/p^nA [Bha21].
- The map $A/p^nA \longrightarrow A^+/p^nA^+$ is integral.
- Apply the application theorem.

This includes results in mixed characteristic, not just char. p.

Conclusion

1. HMCM is not preserved under localization in general.
2. CCM is stable under localization, and CCM $\Rightarrow$ locally HMCM (A.).
3. Integral, locally balanced big CM algebras are CCM (A.).



In particular, A^+ **and** A^+/p^nA^+ **are locally HMCM**.

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Backup: perfection and weak F-nilpotence (A.)

Let A be a Noetherian ring of characteristic $p > 0$, and put

$$A^{\mathrm{perf}} = \varinjlim \left(A \xrightarrow{F} A \xrightarrow{F} \cdots \right).$$

Theorem 12

The following conditions are equivalent:

1. A *is locally weakly* F*-nilpotent.*
2. A^{perf} *is a locally balanced big CM* A*-algebra.*
3. A^{perf} *is CCM.*

Under these conditions, A^{perf} *is locally HMCM.*

Theorem D of the paper; omitted from the main talk.

Backup: weak proregularity and Koszul homology

Write $\underline{a}^n = (a_1^n, \ldots, a_r^n)$ and let $H_i(\underline{a}^n; A)$ denote Koszul homology. **Weak proregularity** means

$$\forall i > 0 \ \forall n \geq 1 \ \exists m \geq n, \quad H_i(\underline{a}^m; A) \longrightarrow H_i(\underline{a}^n; A) \text{ is the zero map.}$$

In particular, for a single element a this is equivalent to

$$\forall n \geq 1 \ \exists m \geq n, \quad a^{m-n} \operatorname{Ann}_A(a^m) = 0.$$