STUDIES OF HOMOLOGICAL ALGEBRA IN NON-NOETHERIAN CASES

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Abstract. In this paper, we study homological algebra in the category of modules over (not necessarily Noetherian) rings. Section 2 of this paper is mainly based on the author's paper [And21]. In it, we give a simple proof of Schenzel's theorem [Sch03] (Theorem 2.2). This theorem is used in the search for a class of good rings with homological characterisation among non-Noetherian rings.

In section 3, we discuss the class of rings called perfect and consider a generalisation of the minimal free resolution on Noetherian local rings. We also give a simple proof of the sufficient condition part of the theorem by [Cha60] (Theorem 3.2).

1. Preliminaries

Throughout this paper, all rings are commutative with the identity element. In this section, we will discuss the basics of commutative algebra theory. The proofs are omitted if they are given in standard textbooks, for example [AM69], [Mat86] and [BH97].

For a ring A, we denote by $\operatorname{Mod} A$ the category of A-modules and by $\operatorname{mod} A$ the category of finitely generated A-modules.

1.1. **Regular sequences and Cohen–Macaulay rings.**

Let A be a ring and $M \in \operatorname{Mod} A$. $a \in A$ is called M**-regular** if the map $a\cdot : M \to M; x \mapsto ax$ is injective. When $M = A$, it is simply called regular. $\underline{a} = a_1, \dots, a_r \in A$ is called a M**-regular sequence** if for each $1 \leq i \leq r$, a_i is $M/(a_1, \dots, a_{i-1})M$-regular and $M/(a_1, \dots, a_r)M \neq 0$.

Note that when A is Noetherian and $M \in \operatorname{mod} A$, if $\underline{a} = a_1, \dots, a_r \in \operatorname{rad} A$ and $\underline{a}$ is M-regular then any permutation of $\underline{a}$ is also regular.

Proposition 1.1. *Let A be a Noetherian ring and $M \in \operatorname{mod} A$. For an ideal I with $IM \neq M$, lengths of maximal regular sequences contained in I are constant.*

[Mat86] and [BH97] use homological algebra to prove this fact, but it is possible to show this without using it. Here we give an another proof without using homological algebra.

Proof. First, we notice that there is a $P \in \operatorname{Ass} M$ with $I \subset P$ if and only if all elements of I are zero divisors of M, by using the prime avoidance.

Let n be the minimal length among all maximal M-regular sequences contained in I. We use induction on n. This proposition is trivial when $n = 0$.

Consider the case of $n = 1$. Let $a \in I$ be an M-regular element which is maximal as a regular sequence. For each $b \in I$, we will show that every element of I is a zero divisor of M/bM. There is a $P \in \operatorname{Ass} M/aM$ with $I \subset P$ since a is maximal. So there is an $x \in M$ such that $x \notin aM$ and $Ix \subset aM$. Then there is a $y \in M$ with $bx = ay$. If $y \in bM$ then $x \in aM$ since b is M-regular, which is a contradiction. Therefore $y \notin bM$. Now $Ix \subset aM$, then $Iay = Ibx \subset abM$. So

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$Iy \subset bM$ since a is M-regular. Thus all maximal M-regular sequences contained in I have the same length.

If $n > 1$, let $a_1, \dots, a_n \in I$ be a maximal M-regular sequence, $b_1, \dots, b_n \in I$ a M-regular sequence. We denote by $I_i \coloneqq (a_1, \dots, a_i), J_i \coloneqq (b_1, \dots, b_i)$. The following set;

$$\mathscr{P} \coloneqq \bigcup_{0 \leq i \leq n-1} (\operatorname{Ass} M/I_i M \cup \operatorname{Ass} M/J_i M)$$

is finite and each $P \in \mathscr{P}$ satisfies $I \not\subset P$ by the above notice. Thus there is a $c \in I$ such that for all $1 \leq i \leq n-1$, c is M/I_iM and M/J_iM-regular by using the prime avoidance. Now, $a_2, \dots, a_n$ is a maximal M/a_1M-regular sequence whose length is $n-1$. By the hypothesis of induction, $a_2, \dots, a_{n-1}, c$ is a maximal M/a_1M-regular sequence. So M/cM has a maximal sequence whose length is $n-1$. Also $b_1, \dots, b_{n-1}$ is a M/cM-regular sequence since $b_1, \dots, b_{n-1}, c$ is a M-regular sequence. Thus $b_1, \dots, b_{n-1}, c$ is maximal by the hypothesis of induction. Now c is a maximal $M/J_{n-1}M$-regular sequence hence so is b_n. Therefore $b_1, \dots, b_n$ is maximal. □

So we denote by $\operatorname{depth}_I M$ the length of maximal regular sequences contained in I. For a Noetherian local ring $(A, \mathfrak{m})$, $\operatorname{depth}_{\mathfrak{m}} M$ is simply written as $\operatorname{depth} M$. Note that $\operatorname{depth} M \leq \dim M$ if $(A, \mathfrak{m})$ is Noetherian local and $M \in \operatorname{mod} A$.

Definition 1.2. Let A be a Noetherian ring. $M \in \operatorname{mod} A$ is said to be **Cohen–Macaulay** if for each $P \in \operatorname{Spec} A$, $\operatorname{depth} M_P = \dim M_P$. A is called Cohen–Macaulay if A is Cohen–Macaulay as an A-module.

One of the reasons why Cohen–Macaulay modules have been well studied is that, their dimensions are given by homological data by below Rees's theorem.

Theorem 1.3 (Rees). *Let A be a Noetherian ring, $M \in \operatorname{mod} A$ and I ideal with $IM \neq M$. Then;*

$$\operatorname{depth}_I M = \inf \left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i(A/I, M) \neq 0 \right\}.$$

Proof. [Mat86, Theorem 16.7], [BH97, Theorem 1.2.5]. □

The presice definition of the Ext functor will be reviewd in Definition 1.10. In order to define Gorenstein rings, in the following subsections, we will outline some basic knowledges about homological theory of commutative rings.

1.2. δ-functors.

We used the Ext functor above, which is one example of what is called a δ-functor. It is introduced by [Gro57] and mentioned in [Har77] without the proof. The theory of δ-functors is very powerful to show that some family of functors coincides with some derived functors. Some textbooks, e.g. [Mat86, §18, lemma 2] and [BH97, Theorem 3.5.6], give proofs by using δ-functor implicitly. We will summarise it here.

Let $\mathscr{A}$ be a category. $\mathscr{A}$ is called an **Abelian category** if the following conditions are satisfied.

(1) $\mathscr{A}$ has the zero object.
(2) For each $A, B \in \mathscr{A}$, $\operatorname{Hom}_{\mathscr{A}}(A, B)$ has a natural Abelian group structure.
(3) For each $A, B \in \mathscr{A}$, there is the product $A \times B$.
(4) For each morphism f, there is the kernel and the cokernel.
(5) For each morphism f, the canonical morphism from $\operatorname{Coim} f$ to $\operatorname{Im} f$ is an isomorphism.

Homological Algebra theory is developed on Abelian categories. See a textbook such as [Nor60] for basic information on the chain complex and the definition of homology, cohomology.

An Abelian category $\mathscr{A}$ is said to **have enough injectives** if for any $A \in \mathscr{A}$ there is an injective object $I \in \mathscr{A}$ and an injection $\varepsilon : A \to I$. Similarly, $\mathscr{A}$ is said to **have enough projectives** when for each $A \in \mathscr{A}$ there is a projective object $P \in \mathscr{A}$ and a surjection $\varepsilon : P \to A$.

Let A be a ring. $\operatorname{Mod} A$ is a typical example of an Abelian category. This has enough projectives (by taking free modules) and we can show that it also has enough injectives (by using the Pontrjagin dual).

Definition 1.4. Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. Suppose $\mathscr{A}$ has enough injectives. Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive left exact functor. $\boldsymbol{I}^\bullet$ denotes an injective resolution of $A \in \mathscr{A}$. The functor;

$$R^iF\colon \mathscr{A} \to \mathscr{B}; A \mapsto H^i(F(I^\bullet))$$

is called a **right derived functor** of F.

Note that derived functors are independent up to natural transformations of the choice of an injective resolution. The following is a characteristic property of the derived functor.

Proposition 1.5. *Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, and suppose $\mathscr{A}$ has enough injectives. Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive left exact functor. Then*

(1) $R^0F \cong F$ *(as functors).*

(2) *For any exact sequence in $\mathscr{A}$;*

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

and for each $i \geq 0$, there is a ***connecting morphism*** $\delta^i\colon R^iF(A_3) \to R^{i+1}F(A_1)$ *such that;*

$$0 \longrightarrow F(A_1) \xrightarrow{F(f)} F(A_2) \xrightarrow{F(g)} F(A_3) \xrightarrow{\delta^0} \cdots$$
$$\xrightarrow{\delta^{i-1}} R^iF(A_1) \xrightarrow{R^iF(f)} R^iF(A_2) \xrightarrow{R^iF(g)} R^iF(A_3) \xrightarrow{\delta^i} \cdots$$

is an exact sequence in $\mathscr{B}$.

(3) *For given a commutative diagram of the form (where the rows are exact) in $\mathscr{A}$;*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \longrightarrow & 0 \end{array}$$

and for any $i \geq 0$, the following diagram;

$$\begin{array}{ccc} R^iF(A_3) & \longrightarrow & R^{i+1}F(A_1) \\ \downarrow{\scriptstyle R^iF(\gamma)} & & \downarrow{\scriptstyle R^{i+1}F(\alpha)} \\ R^iF(B_3) & \longrightarrow & R^{i+1}F(B_1) \end{array}$$

is commutative in $\mathscr{B}$.

(4) *For each injective object $I \in \mathscr{A}$ and for any $i > 0$, $R^iF(I) = 0$.*

Proof. See [CE56, Chap. V, §4]. □

The δ-functor can be thought of as an extract of the above property.

Definition 1.6. Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. A family of additive functors $T^\bullet := \{T^i\}$ is called a **δ-functor** if the following conditions hold;

(1) For any exact sequence in $\mathscr{A}$;

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

and for each $i \geq 0$, there are **connecting morphisms** $\delta^i \colon T^i(A_3) \to T^{i+1}(A_1)$ such that;

$$\begin{aligned} 0 \longrightarrow T^0(A_1) &\xrightarrow{T^0(f)} T^0(A_2) \xrightarrow{T^0(g)} T^0(A_3) \xrightarrow{\delta^0} \cdots \\ \xrightarrow{\delta^{i-1}} T^i(A_1) &\xrightarrow{T^i(f)} T^i(A_2) \xrightarrow{T^i(g)} T^i(A_3) \xrightarrow{\delta^i} \cdots \end{aligned}$$

is an exact sequence in $\mathscr{B}$.

(2) Given a commutative diagram in $\mathscr{A}$ of the form (where the rows are exact);

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \longrightarrow & 0 \end{array}$$

and for any $i \geq 0$ the following diagram;

$$\begin{array}{ccc} T^i(A_3) & \longrightarrow & T^{i+1}(A_1) \\ \downarrow{\scriptstyle T^i(\gamma)} & & \downarrow{\scriptstyle T^{i+1}(\alpha)} \\ T^i(B_3) & \longrightarrow & T^{i+1}(B_1) \end{array}$$

is commutative in $\mathscr{B}$.

We define that two δ-functors are isomorphic in the following way. Let $T^\bullet, U^\bullet$ be δ-functors. A family of natural transformations $\theta^\bullet = \{\theta^i \colon T^i \Rightarrow U^i\}$ is called a **morphism of δ-functors** if for each exact sequence ;

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

in $\mathscr{A}$, the following diagram;

$$\begin{array}{ccc} T^i(A_3) & \xrightarrow{\delta^i_T} & T^{i+1}(A_1) \\ \downarrow{\scriptstyle \theta^i_{A_3}} & & \downarrow{\scriptstyle \theta^{i+1}_{A_1}} \\ U^i(A_3) & \xrightarrow{\delta^i_U} & U^{i+1}(A_1) \end{array}$$

is commutative. An isomorphism is a morphism which has a two-sided inverse.

Definition 1.7. Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. The δ-functor $T^\bullet$ is called **universal** if for each δ-functor $U^\bullet$ and a natural transformation $\theta \colon T^0 \Rightarrow U^0$, there is a unique morphism of δ-functors $\theta^\bullet \colon T^\bullet \to U^\bullet$ such that $\theta^0 = \theta$.

By the definition, two universal δ-functors such that $T^0 = U^0$ are isomorphic up to the unique isomorphism. So for each additive functor $F \colon \mathscr{A} \to \mathscr{B}$, if a universal δ-functor $T^\bullet$ with $T^0 = F$ exists, it is unique up to unique isomorphism. A universal δ-functor $T^\bullet$ with this property is called a right satellite functor of F.

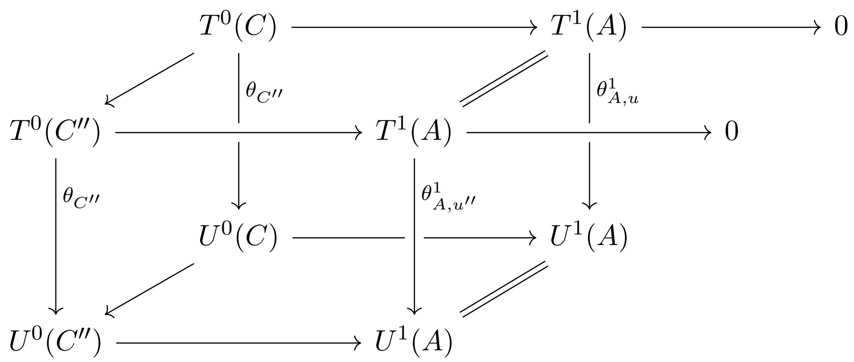
The following property gives a sufficient condition for a δ-functor to be universal, which shows that the derived functor is also universal if $\mathscr{A}$ has enough injectives.

Definition-Proposition 1.8. *Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, F an additive functor. F is said to be* ***effaceable*** *if for each $A \in \mathscr{A}$, there is an $M \in \mathscr{A}$ and an injection (monomorphism) $u\colon A \to M$ such that $F(u) = 0$. Under these terminologies, a δ-functor $T^\bullet$ is universal if for each $i > 0$, T^i is effaceable.*

Proof. Let $U^\bullet$ be a δ-functor and $\theta\colon T^0 \Rightarrow U^0$ a natural transformation. We show that there exists uniquely a morphism of δ-functors $\theta^\bullet\colon T^\bullet \to U^\bullet$ such that $\theta^0 = \theta$. We construct it inductively. For any $A \in \mathscr{A}$, there is an injection $u\colon A \to M$ such that $T^1(u) = 0$ since T^1 is effaceable. Let C be the cokernel of u. We consider the long exact sequences induced by $0 \longrightarrow A \xrightarrow{u} M \xrightarrow{\pi} C \longrightarrow 0$. So we get the following commutative diagram.

$$\begin{array}{ccccccc}
T^0(M) & \xrightarrow{T^0(\pi)} & T^0(C) & \xrightarrow{\delta_T^0} & T^1(A) & \xrightarrow{T^1(u)=0} & 0 \\
\downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_C} & & \downarrow & & \\
U^0(M) & \xrightarrow{U^0(\pi)} & U^0(C) & \xrightarrow{\delta_U^0} & U^1(A) & &
\end{array}$$

Now $\theta^1_{A,u} := \delta^0_U \circ \theta_C \circ (\delta^0_T)^{-1}\colon T^1(A) \to U^1(A)$ is well-defined since the rows are exact. We show $\theta^1_{A,u}$ is independent of the choice of u. Let $u'\colon A \to M'$ be an injection such that $T^1(u') = 0$. $M \sqcup_A M'$ denote the cofibre product of M and M' on A. Then we get an injection $u''\colon A \to M \sqcup_A M'$ such that $T^1(u'') = 0$. Let C'' be the cokernel of u''. Then the following diagram;



is commutative. So we have $\theta^1_{A,u} = \theta^1_{A,u''}$, similarly we obtain $\theta^1_{A,u'} = \theta^1_{A,u''}$, then we get $\theta^1_{A,u} = \theta^1_{A,u'}$. So θ^1_A is independent of the choice of u.

Secondly, we show that for each $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ the following diagram;

$$\begin{array}{ccc}
T^1(A) & \xrightarrow{T^1(f)} & T^1(B) \\
\downarrow{\scriptstyle \theta^1_A} & & \downarrow{\scriptstyle \theta^1_B} \\
U^1(A) & \xrightarrow{U^1(f)} & U^1(B)
\end{array}$$

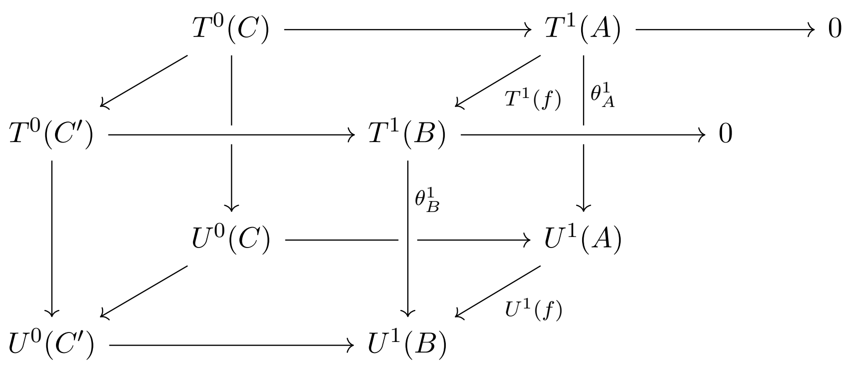
is commutative to prove θ^1 is a natural transformation. For any injections $u\colon A \to M$ and $v\colon B \to N$ with $T^1(u) = T^1(v) = 0$, we take the cofibre product;

$$\begin{CD} A @>{u}>> M \\ @V{v\circ f}VV @VVV \\ N @>{u'}>> M \sqcup_A N \end{CD}$$

then u' is injective. So we have an injection $u' \circ v\colon B \to M \sqcup_A N$ with $T^1(u' \circ v) = 0$. Then we replace N by $M \sqcup_A N$ and get the following commutative diagram with exact rows;

$$\begin{CD} 0 @>>> A @>>> M @>>> C @>>> 0 \\ @. @V{f}VV @VVV @VVV @. \\ 0 @>>> B @>>> N @>>> C' @>>> 0. \end{CD}$$

So θ^1 is a natural transformation since the following diagram;



is commutative.

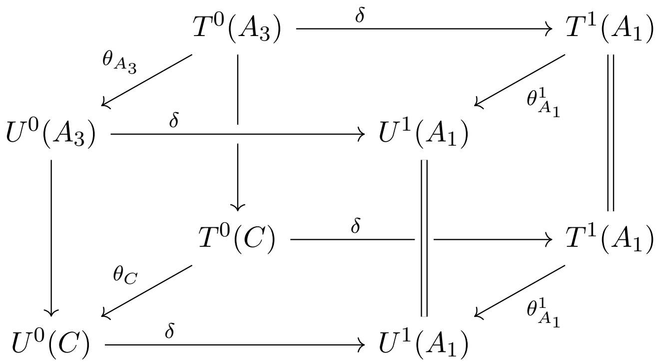
Finally, we show that θ^1_A is commutative with the connecting morphisms. Let

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

be a short exact sequence in $\mathscr{A}$, we use the same method as above for the injection $u\colon A_1 \to M$ with $T^1(u) = 0$ so that each row of the following commutative diagram is exact.

$$\begin{CD} 0 @>>> A_1 @>>> A_2 @>>> A_3 @>>> 0 \\ @. @| @VVV @VVV @. \\ 0 @>>> A_1 @>>> M @>>> C @>>> 0 \end{CD}$$

We consider the following diagram.



The desired commutativity of $\theta^1_{A_1}, \theta_{A_3}$ and δ follows from commutativity of other squares, which follow from the construction of $\theta^1_{A_1}$ and the facts that $T^\bullet$ and $U^\bullet$ are δ-functors and that θ is a natural transformation.

This shows that θ^1 is a natural transformation commutative with connecting morphisms, and its uniqueness can be seen from its construction (the universality of cokernel). In this way, θ^{i+1} can be constructed from θ^i inductively on i. $\square$

Corollary 1.9. *Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, and suppose $\mathscr{A}$ has enough injectives. Let $T^\bullet\colon \mathscr{A} \to \mathscr{B}$ be a universal δ-functor, then T^0 is left-exact and for each $i \geq 0$ there is a natural isomorphism $T^i \cong R^iT^0$.*

Proof. T^0 is left exact by the definition of a δ-functor, so there are right derived functors R^iT^0. For each $i > 0$, R^iT^0 is effaceable by (4) of Proposition 1.5, then $R^\bullet T^0$ is an universal δ-functor. Now $R^0T^0 = T^0$, so there is a unique isomorphism $R^\bullet T^0 \cong T^\bullet$ by universality. $\square$

Although we discussed the cochain complex here, we can make a similar argument for a left derived functor by considering a chain complex.

As typical examples of derived functors, we define Tor and Ext.

Definition 1.10. For an $M \in \operatorname{Mod} A$, the functor $M \otimes -$ is right exact. We denote by $\operatorname{Tor}_\bullet(M, -)$ the left derived functor of this. Also, we define $\operatorname{Ext}^\bullet(M, -)$ to be the right derived functor of the left exact functor $\operatorname{Hom}(M, -)$.

Ext groups may also be obtained as the derived functor of $\operatorname{Hom}(-, N)$, but by computing the double chain complex we see that these two definitions coincide.

1.3. **Homological dimensions.**

The concept of dimensions defined by homological information such as the projective dimension and global dimension is collectively called a **homological dimension**.

Definition 1.11. Let A be a ring. For an $M \neq 0 \in \operatorname{Mod} A$, we denote by $\operatorname{prj.dim}_A M$ the minimal length among all projective resolutions of M. Similarly, we define $\operatorname{inj.dim}_A M$ to be the minimal length among all injective resolutions of M. For $M = 0$, we define $\operatorname{prj.dim} M = \operatorname{inj.dim} M = -\infty$.

Note that for each $M \in \operatorname{Mod} A$ and $n \geq 0$;

$$\operatorname{prj.dim} M \leq n \iff \text{For all } N \in \operatorname{Mod} A, \operatorname{Ext}^{n+1}(M, N) = 0,$$

$$\text{inj.dim}\, M \leq n \iff \text{For all } N \in \text{Mod}\, A, \text{Ext}^{n+1}(N, M) = 0.$$

The following theorem is well known and very powerful.

Theorem 1.12 (Auslander–Buchsbaum)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. For an $M \neq 0 \in \text{mod}\, A$ with $\text{prj.dim}\, M < \infty$;*

$$\text{prj.dim}\, M + \text{depth}\, M = \text{depth}\, A.$$

Proof. [Mat86, Theorem 19.1] or [BH97, Theorem 1.3.3]. □

Definition 1.13. A Noetherian local ring $(A, \mathfrak{m})$ is called a **Gorenstein ring** if $\text{inj.dim}\, A < \infty$. For an arbitrary Noetherian ring A, A is called Gorenstein if for each $P \in \text{Spec}\, A, A_P$ is Gorenstein.

The second half of the definition is justified by the following lemma.

Lemma 1.14. *Let A be a Noetherian ring. For each multiplicatively closed subset $S \subset A$ and an injective A-module E, E_S is an injective A_S-module.*

Proof. [Mat86, §18, lemma 5.]. □

We note that a Gorenstein ring is Cohen–Macaulay by the following theorem.

Theorem 1.15. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with $\dim A = d$. The following are equivalent.*

(1) *A is Gorenstein, i.e. $\text{inj.dim}\, A < \infty$.*
(2) $\text{inj.dim}\, A = d$.
(3) $\text{Ext}^d(k, A) \cong k$ *and for each* $i \neq d, \text{Ext}^i(k, A) = 0$.
(4) *A is Cohen–Macaulay and $\text{Ext}^d(k, A) \cong k$.*
(5) *A is Cohen–Macaulay and all ideals generated by a parameter system are irreducible.*
(6) *A is Cohen–Macaulay and there is an ideal generated by a parameter system which is irreducible.*
(7) *There is an $i > d$ with $\text{Ext}^i(k, A) = 0$.*

Proof. [Mat86, Theorem 18.1]. □

2. WEAKLY PROREGULAR SEQUENCE

In this section we present the theory of weakly proregular sequence following [And21].

2.1. Overview.

Let A be a ring and I an ideal of A. The functor Γ_I is defined by;

$$\Gamma_I(M) \coloneqq \{x \in M \mid I^n x = 0 \text{ for some } n \geq 0\}$$

for an $M \in \text{Mod}\, A$. Then, the local cohomology functors $H^i_I(-)$ are defined as the right derived functors of $\Gamma_I(-)$. In Noetherian cases, the local cohomology can be written by using the Čech cohomology as follows.

Theorem 2.1. *Let A be a Noetherian ring, $\underline{a} = a_1, \dots, a_r$ a sequence of elements of A and $I = (a_1, \dots, a_r)$. $\check{H}^i(\underline{a}, M)$ denotes the Čech cohomology (see Definition 2.3). Then, there are isomorphisms;*

$$H^i_I(M) \cong \check{H}^i(\underline{a}, M) \qquad (*)$$

for any $M \in \text{Mod}\, A$.

See [BH97, Theorem 3.5.6.] for the proof. This result was generalised by [Sch03]. For an arbitrary ring A and a sequence $\underline{a} = a_1, \ldots, a_r$ with $I = (a_1, \ldots, a_r)$, he showed that formula $(*)$ is true for any $M \in \operatorname{Mod} A$ if and only if $\underline{a}$ is a weakly proregular sequence in the following sence. Let $H_i(\underline{a})$ be the Koszul homology of the sequence $\underline{a}$. $\underline{a} = a_1, \ldots, a_r$ is called a weakly proregular sequence if for any $1 \leq i \leq r$ and for each $n > 0$ there is an $m \geq n$ such that the natural map

$$H_i(\underline{a}^m) \to H_i(\underline{a}^n)$$

is the zero map, where $\underline{a}^n$ is the sequence defined by $a_1^n, \ldots, a_r^n$.

The goal of this section is to explain the following result without using notions of derived category theory.

Theorem 2.2 ([Sch03, Theorem 3.2]). *Let A be a ring, $\underline{a} = a_1, \ldots, a_r$ a sequence of elements of A and $I = (a_1, \ldots, a_r)$. $\underline{a}$ is a weakly proregular sequence if and only if for any i and $M \in \operatorname{Mod} A$, $H^i_I(M) \cong \check{H}^i(\underline{a}, M)$ functorially on M.*

In subsection 2.2 we review the definition of Čech cohomology in commutative algebra. In subsection 2.3 we present the theory of weakly proregular sequences, following [Sch03, Sect. 2]. As for the weakly proregular sequence and Koszul homology, [PSY14, Sect. 4] also obtains some results using derived categories with a different approach from [And21]. Finally, we prove Theorem 2.2 in subsection 2.4.

2.2. **Čech cohomology and Koszul complex.**

In this subsection, we review the Čech cohomology of rings and modules. Let A be a ring and fix a sequence $a_1, \ldots, a_r$ of elements of A. For each $I = \{j_1, \ldots, j_i\} \subset \{1, \ldots, r\}$ $(j_1 < \cdots < j_i)$, let $a_I = a_{j_1} \ldots a_{j_i}$. $e_1, \ldots, e_r$ denotes the standard basis of A^r and let $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$.

Definition 2.3. Let A be a ring, $\underline{a} = a_1, \ldots, a_r \in A$. For each $1 \leq i \leq r$, $C^i(\underline{a})$ is the module defined by the following equation

$$C^i(\underline{a}) \coloneqq \sum_{\#I=i} A_{a_I} e_I.$$

Then we define $C^\bullet(\underline{a})$ to be the complex defined by the following differentials

$$d^i \colon C^i(\underline{a}) \to C^{i+1}(\underline{a}); e_I \mapsto \sum_{j=1}^{n} e_I \wedge e_j.$$

It is called a **Čech complex**. $\check{H}^i(\underline{a})$ denotes the cohomology of this complex and it is called a **Čech cohomology**.

For an $M \in \operatorname{Mod} A$, we define $C^\bullet(\underline{a}, M) \coloneqq C^\bullet(\underline{a}) \otimes M$. Here $\check{H}^i(\underline{a}, M)$ denotes the cohomology of $C^\bullet(\underline{a}, M)$.

Proposition 2.4. *Let A be a ring. For each $\underline{a} = a_1, \ldots, a_r \in A$, $\check{H}^\bullet(\underline{a}, -) = \{\check{H}^i(\underline{a}, -)\}_{i \geq 0}$ constitutes a δ-functor.*

Proof. Consider an exact sequence of A-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \ .$$

Since $C^\bullet(\underline{a}, -) = C^\bullet(\underline{a}) \otimes -$ and each component of the Čech complex is a flat A-module, the following sequence of complexes is exact;

$$0 \longrightarrow C^\bullet(\underline{a}, M_1) \longrightarrow C^\bullet(\underline{a}, M_2) \longrightarrow C^\bullet(\underline{a}, M_3) \longrightarrow 0$$

then there are connecting morphisms. So $\check{H}^\bullet(\underline{a}, -)$ is a δ-functor. □

For the result we want, we need to look at the relationship between Čech complex and Koszul complex.

For $\underline{a} = a_1, \ldots, a_r \in A$, let $\{e_i\}$ be the standard basis of a free A-module A^r. $f\colon A^r \to A; e_i \mapsto a_i$ induces a chain complex $K_\bullet(\underline{a})$. In other words, $K_\bullet(\underline{a})$ is the complex defined by following equations;

$$K_i(\underline{a}) := \bigwedge^i A^r,$$

$$d_i\colon K_i(\underline{a}) \to K_{i-1}(\underline{a}); x_1 \wedge \cdots \wedge x_i \mapsto \sum_{j=1}^{i} (-1)^{j+1} f(x_j) x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_i.$$

Note that $K_\bullet(\underline{a})$ does not depend on the order of a_i .

We get a co-chain complex $K^\bullet(\underline{a})$ via the contravariant functor $\mathrm{Hom}(-, A)$;

$$K^\bullet(\underline{a})\colon\ 0 \longrightarrow A \longrightarrow \mathrm{Hom}(K_1(\underline{a}), A) \longrightarrow \cdots .$$

$K^\bullet(\underline{a})$ is called a Koszul complex. For each $M \in \mathrm{Mod}\, A$, $K^\bullet(\underline{a}, M) = K^\bullet(\underline{a}) \otimes M$. We denote by $H^i(\underline{a}, M)$ the cohomology of Koszul complex.

Lemma 2.5. *Let A be a ring and $\underline{a} = a_1, \ldots, a_r \in A$. For each $1 \le i \le r$;*

$$\varphi^i\colon K^i(\underline{a}) \to C^i(\underline{a}); (e_I)^* \mapsto (1/a_I) e_I$$

is a morphism of complexes.

Proof. Let δ^i be the differential of the Koszul complex. Then;

$$\delta^i(e_I^*)(e_J) = \begin{cases} a_j & (j \notin I, J = I \cup \{j\}) \\ 0 & (\text{otherwise}) \end{cases}$$

So

$$\varphi^{i+1} \circ \delta^i(e_I^*) = \sum_{j \notin I} \frac{a_j}{a_I a_j} e_I \wedge e_j = \sum_{j \notin I} \frac{1}{a_I} e_I \wedge e_j$$

is equal to $d^i \circ \varphi^i(e_I^*)$. □

For any pair $n \le m$, we set

$$\varphi^\bullet_{mn}\colon K^\bullet(\underline{a}^n) \to K^\bullet(\underline{a}^m); (e_I)^* \mapsto (a_I)^{m-n}(e_I)^*$$

then $\{K^\bullet(\underline{a}^n), \varphi^\bullet_{mn}\}_{n \in \mathbb{N}}$ is an inductive system.

Proposition 2.6. *Let A be a ring, $a_1, \ldots, a_r \in A$. Then;*

$$\varinjlim K^\bullet(\underline{a}^n) \cong C^\bullet(\underline{a}).$$

Proof. We define $\varphi_n^\bullet\colon K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a}^n) = C^\bullet(\underline{a})$ in the same way the an above lemma. Then $\varphi_m^\bullet \circ \varphi_{nm}^\bullet = \varphi_n^\bullet$ where $n \leq m$. So we have $\varphi\colon \varinjlim K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a})$. Each element of $C^i(\underline{a})$ is represented by a finite sum of $(b_I/a_I^{n_I})e_I$, so it can be displayed as $\sum(1/a_I^n)b_I e_I$ by taking the maximum of $n = \max\{n_I\}$ and replacing b_I. Then it is the image of $\sum(b_I e_I) \in K^i(\underline{a}^n)$, so φ is surjective.

Secondly, we show φ is injective. Assume $\varphi_n^i(x) = 0$ for $x \in K^i(\underline{a}^n)$. If $x = \sum b_I e_I^*$ then $\varphi_n^i(x) = \sum(b_I/a_I^n)e_I = 0$, so $b_I/a_I^n = 0$ in $A_{a_I^n}$. Therefore if we take a sufficiently large l, $a_I^l b_I = 0$. So $\varphi_{ln}^i(x) = 0$ by increasing l thus φ is injective. □

Since the functor of taking the inductive limit is exact, the following corollary follows.

Corollary 2.7. *Let A be a ring, $a_1, \dots, a_r \in A$. For each $M \in \operatorname{Mod} A$;*

$$\check{H}^i(\underline{a}, M) \cong \varinjlim H^i(\underline{a}^n, M).$$

2.3. **Weakly proregular sequences.**

In this subsection, we summarise the weakly proregular sequence following [Sch03, Sect.2].

Definition 2.8. Let $\mathscr{A}$ be an Abelian category, (X_n, φ_{mn}) a projective system in $\mathscr{A}$. (X_n) is said to be **essentially zero** or **pro-zero** if for each n, there is an $m \geq n$ such that $\varphi_{mn}\colon X_m \to X_n$ is the zero map.

Obviously, if (X_n) is essentially zero then $\varprojlim X_n = 0$.

Proposition 2.9 ([Gro66, §2, Remark 2])**.** *Let $\mathscr{A}$ be an Abelian category. We consider an exact sequence of projective systems in $\mathscr{A}$;*

$$0 \longrightarrow (X_n) \longrightarrow (Y_n) \longrightarrow (Z_n) \longrightarrow 0\ .$$

Then (Y_n) is essentially zero if and only if the other two are essentially zero.

Proof. If (Y_n) is essentially zero, then it is clear that the other two are so. We show the opposite. For each n, there is an $m \geq n$ such that $X_m \to X_n$ is the zero map since (X_n) is essentially zero. Similarly there is an $l \geq m$ such that $Z_l \to Z_m$ is the zero map, then we have the following commutative diagram with the exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_l & \longrightarrow & Y_l & \longrightarrow & Z_l & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi_{lm}} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & X_m & \longrightarrow & Y_m & \longrightarrow & Z_m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi_{mn}} & & \downarrow & & \\
0 & \longrightarrow & X_n & \longrightarrow & Y_n & \longrightarrow & Z_n & \longrightarrow & 0
\end{array}$$

So we get $\varphi_{ln} = \varphi_{mn} \circ \varphi_{lm} = 0$ by an easy diagram chasing. □

We use same symbols as before for the Koszul and Čech complexes. Note that $(K_i(\underline{a}^n))_{n\in\mathbb{N}}$ is a projective system defined by $K_i(\underline{a}^m) \to K_i(\underline{a}^n); e_I \mapsto a_I^{m-n}e_I\ (m \geq n)$.

Definition 2.10. Let A be a ring. $\underline{a} = a_1, \dots, a_r \in A$ is called a **weakly proregular sequence** if for each $1 \leq i \leq r$, the projective system $\{H_i(\underline{a}^n)\}$ is essentially zero.

The property of being weakly proregular does not depend on the order of $\underline{a}$ by the definition.

Proposition 2.11. *Let A be a ring, $\underline{a} = a_1, \dots, a_r \in A$. $\underline{a}$ is a weakly proregular sequence if and only if $\check{H}^i(\underline{a}, -)$ is an effaceable functor for $i > 0$.*

Proof. Assume that $\underline{a}$ is a weakly proregular sequence. Let I be an injective module. Now there is an isomorphism $H^i(\underline{a}^n, I) \cong \operatorname{Hom}(H_i(\underline{a}^n), I)$ since $K^\bullet(\underline{a}^n, I) = \operatorname{Hom}(K_\bullet(\underline{a}^n), I)$ and $\operatorname{Hom}(-, I)$ is an exact functor. For each $n \geq 0$, there is an $m \geq n$ such that $H_i(\underline{a}^m) \to H_i(\underline{a}^n)$ is the zero map since $H_i(\underline{a}^n)$ is essentially zero. So $\check{H}^i(\underline{a}, I) = \varinjlim H^i(\underline{a}^n, I) = 0$.

Secondly, assume that $\check{H}^i(\underline{a}, -)$ is an effaceable functor for $i > 0$. For each $n \geq 0$, we have an injective module I and an injection $\varepsilon \colon H_i(\underline{a}^n) \to I$. Then there is an $m \geq n$ such that;

$$H_i(\underline{a}^m) \longrightarrow H_i(\underline{a}^n) \xrightarrow{\ \varepsilon\ } I$$

is the zero map by $\varepsilon \in H^i(\underline{a}^n, I)$ and $\varinjlim H^i(\underline{a}^n, I) = 0$. □

From Corollary 1.9, Čech cohomology is the derived functor of $\check{H}^0(\underline{a}, -)$ if $\underline{a}$ is a weakly proregular sequence. So the next question of interest is when is a sequence weakly proregular. We introduce proregular sequences by [GM92], and prove that every sequence $\underline{a}$ is weakly proregular in the Noetherian case.

Definition 2.12. Let A be a ring, $\underline{a} = a_1, \dots, a_r \in A$. $\underline{a}$ is called a **proregular sequence** if for each $1 \leq i \leq r$ and $n > 0$, there is an $m \geq n$ such that $((a_1^m, \dots, a_{i-1}^m) : a_i^m A) \subset ((a_1^n, \dots, a_{i-1}^n) : a_i^{m-n} A)$.

Note that a regular sequence is proregular.

Proposition 2.13 ([Sch03, Sect. 2])**.** *Let A be a Noetherian ring. For each $\underline{a} = a_1, \dots, a_r \in A$, $\underline{a}$ is a proregular sequence.*

Proof. Let $J_m := ((a_1^m, \dots, a_{i-1}^m) : a_i^m A), I_{n,m} := ((a_1^n, \dots, a_{i-1}^n) : a_i^{m-n} A)$. For each n, $\{I_{n,m}\}_{m \geq n}$ is an ascending chain of ideals, hence there is an $m_0 \geq n$ such that for each $m \geq m_0, I_{n,m_0} = I_{n,m}$. Let $m := m_0 + n$, then for each $a \in J_{m_0}$, $aa_i^{m-n} = aa_i^{m_0} \in (a_1^{m_0}, \dots, a_{i-1}^{m_0}) \subset (a_1^n, \dots, a_{i-1}^n)$. So $a \in I_{n,m} = I_{n,m_0}$. □

Proposition 2.14 ([Sch03, lemma 2.7])**.** *Let A be a ring. A proregular sequence is weakly proregular.*

Proof. We use induction on r. When $r = 1$, let $a \in A$ be proregular. Then for each $n > 0$, there is an $m \geq n$ such that $\operatorname{Ann} a^m \subset \operatorname{Ann} a^{m-n}$. So $(H_1(a^n))$ is essentially zero since $H_1(a^n) = \operatorname{Ann} a^n$. Now we assume that claim holds up to $r - 1$. For $\underline{a} = a_1, \dots, a_r$ and $\underline{a}' = a_1, \dots, a_{r-1}$, the exact sequence of complexes;

$$0 \longrightarrow K_\bullet(\underline{a}'^n) \longrightarrow K_\bullet(\underline{a}^n) \longrightarrow K_\bullet(\underline{a}'^n)(-1) \longrightarrow 0$$

induces the exact sequence of homology;

$$\cdots \longrightarrow H_i(\underline{a}'^n) \xrightarrow{(-1)^i a_r^n}$$

$$H_i(\underline{a}'^n) \longrightarrow H_i(\underline{a}^n) \longrightarrow H_{i-1}(\underline{a}'^n) \xrightarrow{(-1)^{i-1} a_r^n}$$

$$H_{i-1}(\underline{a}'^n) \longrightarrow \cdots$$

hence we have the following exact sequence;

$$0 \longrightarrow H_0(a_r^n, H_i(\underline{a}'^n)) \longrightarrow H_i(\underline{a}^n) \longrightarrow H_1(a_r^n, H_{i-1}(\underline{a}'^n)) \longrightarrow 0$$

and this induces the exact sequence of projective systems. The first projective system is essentially zero by the assumption of induction. Also for each $i > 1$, the third system is essentially zero since $H_1(a_r^n, H_{i-1}(\underline{a}'^n)) = \{x \in H_{i-1}(\underline{a}'^n) \mid a_r^n x = 0\}$. If $i = 1$, the system with ;

$$H_1(a_r^n, H_0(\underline{a}'^n)) = \left\{x \in H_0(\underline{a}'^n) \;\middle|\; a_r^n x = 0\right\}$$

is essentially zero since $\underline{a}$ is proregular. So this completes the proof by the induction. □

Corollary 2.15. *Let A be a Noetherian ring. For each $\underline{a} = a_1, \dots, a_r \in A$, $\underline{a}$ is weakly proregular.*

2.4. **Local cohomology.**

Let A be a ring and I an ideal of A. The functor $\Gamma_I(-)$ is defined by;

$$\Gamma_I(M) \coloneqq \{x \in M \mid I^n x = 0 \text{ for some } n \geq 0\}$$

for an $M \in \operatorname{Mod} A$. Note that $\Gamma_I(M) = \varinjlim \operatorname{Hom}_A(A/I^n, M)$ and this isomorphism is functorial in M. By the definition, Γ_I is a left exact functor.

Definition 2.16. Let A be a ring and I an ideal of A. $H_I^i(-)$ denotes the derived functor of $\Gamma_I(-)$ and it is called a **local cohomology**.

Note that $H^i(M) \cong \varinjlim \operatorname{Ext}^i(A/I^n, M)$.

The definition of the local cohomology ($\Gamma_I(-)$) in the above is standard in commutative algebra (e.g., [BH97]), but differs from the one of [Sch03]. Schenzel defines

$$\Gamma_I(M) \coloneqq \{x \in M \mid \operatorname{Supp}(Ax) \subset V(I)\}.$$

We show that if I is finitely generated, then it coincides with the above definition.

Proposition 2.17. *Let A be a ring, I a finitely generated ideal. For each $M \in \operatorname{Mod} A$;*

$$(*) \qquad \{x \in M \mid I^n x = 0 \textit{ for some } n \geq 0\} = \{x \in M \mid \operatorname{Supp}(Ax) \subset V(I)\}.$$

Proof. Note that $\operatorname{Supp}(Ax) = V(\operatorname{Ann} x)$ since $Ax \cong A/\operatorname{Ann} x$. For $x \in M, V(\operatorname{Ann} x) \subset V(I)$ if and only if $\sqrt{I} \subset \sqrt{\operatorname{Ann} x}$. If there is an $n \geq 0$ such that $I^n x = 0$, then $\sqrt{I} \subset \sqrt{\operatorname{Ann} x}$. So we get that $\subset$ holds in the equation $(*)$. The above holds even if I is not finitely generated, but this assumption is necessary for the inclusion of the reverse. If I is finitely generated, obviously $\sqrt{I} \subset \sqrt{\operatorname{Ann} x}$ implies there is an $n \geq 0$ such that $I^n x = 0$. □

Here is an example where the equality of $(*)$ does not hold if I is not finitely generated.

Example 2.18. Let k be a field. Let $A \coloneqq k[y, x_1, x_2, \dots]/(yx_1, yx_2^2, \dots)$ and $I \coloneqq (x_1, x_2, \dots)$. Now $\operatorname{Supp}(Ay) \subset V(I)$ but there is no $n \geq 0$ such that $I^n y = 0$.

In this paper, we only deal with the local cohomology when I is finitely generated, so there is no problem.

We summarise the relationship between local cohomology and Čech cohomology. First, we note that the 0-th part of each cohomologies are naturally isomorphic.

Lemma 2.19. *Let A be a ring, $\underline{a} = a_1, \dots, a_r \in A$, and $I = (a_1, \dots, a_r)$. For each $M \in \operatorname{Mod} A$;*

$$\Gamma_I(M) \cong \check{H}^0(\underline{a}, M).$$

Proof. Here $\check{H}^0(\underline{a}, M)$ is the kernel of

$$M \to \bigoplus_{i=1}^{r} M_{a_i} e_i; x \mapsto (x/1)e_i.$$

Then for each $x \in \check{H}^0(\underline{a}, M)$ and $1 \leq i \leq r$, there is an $n_i \geq 0$ such that $a_i^{n_i} x = 0$. So we have $x \in \Gamma_I(M)$. Similarly the converse is true, so they are equal as submodules of M. $\square$

With the preparations we have made above, we can prove the results we have been aiming for.

Theorem 2.20 (An elementary proof of [Sch03, Theorem 3.2])**.** *Let A be a ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. Then, $\underline{a}$ is a weakly proregular sequence if and only if for any i and $M \in \operatorname{Mod} A$, $H_I^i(M) \cong \check{H}^i(\underline{a}, M)$ functorially on M.*

Proof. Assume that $\underline{a}$ is weakly proregular. $\check{H}^\bullet(\underline{a}, -)$ is a δ-functor by Proposition 2.4. Moreover $\check{H}^\bullet(\underline{a}, -)$ is universal by Proposition 2.11 and Definition-Proposition 1.8. So $H_I^i(M) \cong \check{H}^i(\underline{a}, M)$ by the above lemma. The converse is true by Proposition 2.11. $\square$

Corollary 2.15 shows that Theorem 2.1, which states that always there are isomorphisms between local cohomologies and Čech cohomologies in the Noetherian case, is a special case of Theorem 2.20.

3. Perfect rings in commutative ring theory

In this section, we discuss **perfectness** in commutative ring theory.

3.1. Overview.

In commutative ring theory, it is essential to calculate the homological dimension to see the homological properties of modules. When A is a Noetherian local ring, any $M \in \operatorname{mod} A$ has the **minimal free resolution**. The length of it equals $\operatorname{prj.dim} M$, therefore it is one of the minimal resolutions ([Mat86, §19, lemma 1]). It is a natural question to ask whether this can be generalised to the case of non-Noetherian rings. As one solution, we prove that any module has a projective resolution whose length is minimal if A is a perfect ring (see Definition 3.10, Proposition 3.11). In subsection 3.1, we present the propositions described above.

In subsection 3.2, we survey the facts about perfect rings mainly according to [Bas60].

In subsection 3.3, we consider a direct product of projective modules and a direct sum of injective modules. In the Noetherian cases, the following theorem is well-known.

Theorem 3.1. *Let A be a ring. A is a Noetherian ring if and only if every direct sum of injective A-modules is again injective.*

This fact is mentioned in [BH97, Remark 3.1.4.] without proof. We give the proof of this proposition in Theorem 3.20. In non-commutative ring theory, Chase ([Cha60]) considered the condition for an infinite direct product. Here, we introduce the case where commutativity is assumed.

Theorem 3.2 ([Cha60, Theorem 3.4])**.** *Let A be a ring. A is an Artinian ring if and only if every product of projective A-modules is again projective.*

It is interesting that in these two theorems, the conditions on the right-hand side have duality in the categorical sense, but the left-hand side conditions don't since an Artinian ring is Noetherian.

The argument in [Cha60] contains a difficult discussion, and in this paper, we try to simplify the proof by adding the assumption that the ring is commutative.

In 3.4 we summarise the previous discussion and discuss the hierarchy of classes of rings around perfectness. It should be noted that the discussions in 3.3 and 3.4 include works in progress, and many of the results are budding and have not yet reached a final conclusion.

3.2. **Minimal projective resolutions.**

First, we introduce the projective cover to define the minimal projective resolution.

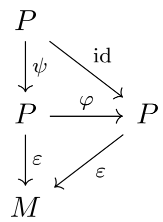
Definition 3.3. Let A be a ring and $M \in \operatorname{Mod} A$. A submodule $N \subset M$ is said to be **superfluous** if for each submodule $L \subset M, L + N = M$ implies $L = M$. Let P be a projective module and $\varepsilon : P \to M$ a surjection. A pair (P, ε) is called a **projective cover** of M if $\ker \varepsilon \subset P$ is superfluous.

We restate the definition of the projective cover in terms of morphisms.

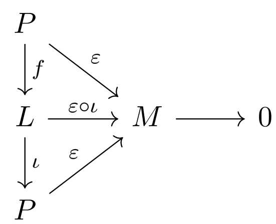
Proposition 3.4. *Let A be a ring, $M \in \operatorname{Mod} A$ and $\varepsilon : P \to M$ a surjection from a projective module P. The followings are equivalent.*

(i) *$\varepsilon : P \to M$ is a projective cover.*
(ii) *For each $N \in \operatorname{Mod} A$ and an A-linear map $\varphi : N \to P$, if $\varepsilon \circ \varphi$ is surjective then φ is a surjection.*
(iii) *For each $\varphi \in \operatorname{Hom}_A(P, P), \varepsilon \circ \varphi = \varepsilon$ implies φ is an isomorphism.*

Proof.
(i) $\Longrightarrow$ (ii)
For each $x \in P$, there is a $y \in N$ with $\varepsilon(\varphi(y)) = \varepsilon(x)$. So $x - \varphi(y) \in \ker \varepsilon$, then $\ker \varepsilon + \operatorname{Im} \varphi = P$. Hence $\operatorname{Im} \varphi = P$ since $\ker \varepsilon$ is a superfluous submodule of P.
(ii) $\Longrightarrow$ (iii)
Since P is projective, there is a $\psi \in \operatorname{Hom}(P, P)$ such that $\varphi \circ \psi = \mathrm{id}_P$. Then, the following diagram



is commutative. So ψ is surjective by (ii). Hence φ and ψ are isomorphisms.
(iii) $\Longrightarrow$ (i)
Let L be a submodule of P such that $L + \ker \varepsilon = P$. For the inclusion $\iota : L \to P, \varepsilon \circ \iota : L \to M$ is surjective. Then, there is an $f : P \to L$ such that the following diagram



is commutative. So $\iota \circ f$ is an isomorphism, and especially ι is surjective. Hence $L = P$. □

Corollary 3.5. *Let A be a ring, $M \in \operatorname{Mod} A$. A projective cover of M is unique if it exists.*

Proof. Suppose M has two different projective covers P, P'. So there is an $f : P' \to P$ such that the following diagram

$$\begin{array}{ccccc} P' & & & & \\ {\scriptstyle f}\,\downarrow & \searrow^{\varepsilon'} & & & \\ P & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \end{array}$$

is commutative. By interchanging P and P' in the above diagram, we see that there is a $g : P \to P'$ and that $\varepsilon' = \varepsilon \circ f, \varepsilon = \varepsilon' \circ g$ holds. Hence $\varepsilon = \varepsilon \circ (f \circ g)$, and $f \circ g$ is an isomorphism by the above proposition, and so is $g \circ f$. Thus $f : P \to P'$ is isomorphic. □

Using the projective cover, we can consider a generalisation of the minimal free resolution.

Definition 3.6. Let A be a ring, $M \in \operatorname{Mod} A$. A projective resolution of M

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is called a **minimal projective resolution** if for each $i \geq 0$, $d_i : P_i \to \ker d_{i-1}$ ($d_0 \coloneqq \varepsilon, d_{-1} \coloneqq 0$) is a projective cover.

Note that a minimal projective resolution is unique if it exists. We prove that it is a projective resolution whose length is minimal.

Lemma 3.7. *Let A be a ring, M an A-module which is not projective, and $\varepsilon : P \to M$ a surjection from a projective module. Now*

$$\operatorname{prj.dim} M = \operatorname{prj.dim} \ker \varepsilon + 1.$$

Proof. We set $n \coloneqq \operatorname{prj.dim} M$, and $K \coloneqq \ker \varepsilon$. For each $N \in \operatorname{Mod} A$, we consider the long exact sequence of Ext induced by the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$;

$$\longrightarrow \operatorname{Ext}^i(P, N) \longrightarrow \operatorname{Ext}^i(K, N) \longrightarrow \operatorname{Ext}^{i+1}(M, N) \longrightarrow \operatorname{Ext}^{i+1}(P, N) = 0 \longrightarrow .$$

For $i > 0$, $\operatorname{Ext}^i(K, N) \cong \operatorname{Ext}^{i+1}(M, N)$ since P is projective. So if $n = \infty$, $\operatorname{prj.dim} K = \infty$.

The remaining case is for $n < \infty$. We can see $\operatorname{prj.dim} K \leq n - 1$ by considering the above exact sequence for $i = n + 1$. Now there is an $N' \in \operatorname{Mod} A$ such that $\operatorname{Ext}^n(M, N') \neq 0$ since $\operatorname{prj.dim} M \not\leq n-1$. So $\operatorname{Ext}^{n-1}(K, N') \neq 0$, then $\operatorname{prj.dim} K \not\leq n-2$. Hence $\operatorname{prj.dim} K = n-1$. □

Proposition 3.8. *Let A be a ring, $M \neq 0$ an A-module. If the minimal projective resolution of M exists and the length of it is n, then* $\operatorname{prj.dim} M = n$.

Proof. This proposition is obvious if M is projective. Suppose that M is not projective. $\operatorname{prj.dim} M = \operatorname{prj.dim} \ker \varepsilon + 1$ by the above lemma. When $n = 1$, $\ker \varepsilon = P_1$ is projective. So $\operatorname{prj.dim} M = 1$. For $n > 1$, $\ker \varepsilon$ is not projective. Therefore, we apply the lemma repeatedly to obtain $\operatorname{prj.dim} M = \operatorname{prj.dim} \ker d_{n-1} + n$. Now $\ker d_{n-1} \cong P_n$, then $\operatorname{prj.dim} M = n$. □

A significant difficulty is that a projective cover may not exist.

Example 3.9. $\mathbb{Z}/n\mathbb{Z}$ does not have a projective cover as a $\mathbb{Z}$-module ($n > 1$).

Note that $\mathbb{Z}/n\mathbb{Z}$ is not projective for $n > 1$. Now, if $\varepsilon : P \to \mathbb{Z}/n\mathbb{Z}$ is a projective cover, then the natural map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a surjection from a projective module, so by Proposition 3.4 there exists a surjection $\mathbb{Z} \to P$. So there is m such that $P \cong \mathbb{Z}/m\mathbb{Z}$. Now $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is not a projective cover because $n\mathbb{Z}$ is not superfluous. So $m > 1$ but it is a contradiction since $\mathbb{Z}/m\mathbb{Z}$ is not projective for $m > 1$. Hence $\mathbb{Z}/n\mathbb{Z}$ does not have a projective cover.

It turns out that $\operatorname{prj.dim} M$ is determined if we can take the minimal projective resolution of M, but the main obstacle is that a projective cover may not exist. As can be seen from the above example, even for PIDs, there is not always a projective cover. However, for finitely generated modules over Noetherian local rings, there is a minimal free resolution which satisfies the definition of a minimal projective resolution.

One (somewhat aggressive) solution is to consider a class of rings such that every module has a projective cover. A ring is called **perfect** if it satisfies this condition. In the next subsection, we give an overview of perfect rings.

3.3. **Perfect rings.**

In this subsection, we review the concept of perfectness in commutative ring theory.

Definition 3.10. Let A be a ring. A is said to be **perfect** if each $M \in \operatorname{Mod} A$ has a projective cover. A is called **semiperfect** if the same condition holds for $\operatorname{mod} A$.

The following proposition follows immediately from the definition.

Proposition 3.11. *Let A be a perfect ring. Then, each $M \in \operatorname{Mod} A$ has the minimal projective resolution.*

Note that even if A is semi-perfect, $M \in \operatorname{mod} A$ may not have the minimal projective resolution. This is because for a projective cover $\varepsilon : P \to M$, $\ker \varepsilon$ may not be finitely generated.

Perfect rings have been studied in the context of non-commutative ring theory. We summarise some terminologies which are often used in non-commutative ring theory but not so often in commutative algebra. For more details, see [Lam01], a textbook on non-commutative ring theory.

First, we define simple modules.

Definition 3.12. Let A be a ring. $M \neq 0 \in \operatorname{Mod} A$ is said to be **simple** if M does not have non-trivial submodules. A direct sum of simple modules is called **semisimple**.

We denote by $\operatorname{Spm} A$ the set of all maximal ideals of A. In non-commutative ring theory, the classification of simple objects is a fundamental problem, but in commutative ring theory, the situation is very "simple".

Proposition 3.13. *Let A be a ring. $M \neq 0 \in \operatorname{Mod} A$ is simple if and only if there is an $\mathfrak{m} \in \operatorname{Spm} A$ such that $M \cong A/\mathfrak{m}$.*

Proof. Suppose M is simple. We take $x \neq 0 \in M$. Now $M = Ax$ since M is simple. Then, the linear map $\varphi : A \to M; a \mapsto ax$ is surjective. So $M \cong A/\ker \varphi$. Here, $\ker \varphi$ is maximal since $A/\ker \varphi \cong M$ is simple. The opposite is trivial. □

Thus a semisimple module over a commutative ring is isomorphic to a direct sum of residue fields. In non-commutative rings, we often define a semilocal ring by using a semisimple ring. That is, a ring A is said to be **semilocal** if $A/\operatorname{rad} A$ is a semisimple ring. Here $\operatorname{rad} A = \bigcap_{\mathfrak{m} \in \operatorname{Spm} A} \mathfrak{m}$ is the Jacobson radical of A. This definition is equivalent to the standard definition in commutative ring theory. That is, the following holds.

Proposition 3.14 ([Lam01, Proposition 20.2])**.** *Let A be a ring.* $\operatorname{Spm} A$ *is a finite set if and only if* $A/\operatorname{rad} A$ *is semisimple.*

Proof. Suppose $\operatorname{Spm} A$ is finite. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the all maximal ideals of A. Any two of them are coprime, so $A/\operatorname{rad} A \cong A/\mathfrak{m}_1 \times \cdots \times A/\mathfrak{m}_r$ by the Chinese remainder theorem. We show the opposite. Suppose $A/\operatorname{rad} A$ is semisimple. By Proposition 3.13, $A/\operatorname{rad} A$ can be written in the following form

$$A/\operatorname{rad} A = \bigoplus_{\mathfrak{m} \in \operatorname{Spm} A} A/\mathfrak{m}.$$

Now $\operatorname{Spm} A$ is a finite set since the left-hand side of above equation is finitely generated. □

Next we will define some terms for an idempotent.

Definition 3.15. Let A be a ring, I an ideal of A. Two idempotents $e, f \in A$ are said to be **orthogonal** if $ef = 0$. An idempotent $e \in A$ is called a **local idempotent** if $Ae \cong \operatorname{End}_A(Ae)$ is a local ring. Also, we say that an idempotent $e + I \in A/I$ **can be lifted modulo** I if there is an idempotent $f \in A$ such that $e + I = f + I$.

Definition 3.16. Let A be a ring. An ideal I is said to be **transfinite nilpotent** if for each sequence $\{a_i\} \subset I$, there is an $n \geq 1$ such that $a_1 \dots a_n = 0$.

We introduce the restatement of perfect rings by [Bas60].

Theorem 3.17 (Bass)**.** *Let A be a ring. The following are equivalent.*

(i) *The ring A is perfect.*
(ii) *The ring A is semilocal and* $\operatorname{rad} A$ *is transfinite nilpotent.*
(iii) *Any direct limit of projective A-modules is projective.*
(iv) *The ring A satisfies the descending chain condition on principal ideals.*

Proof. See [Bas60, Theorem P]. □

For a semiperfect ring, the following characterisation is studied by [Bas60] and [Mül70].

Theorem 3.18 (Bass–Müller)**.** *Let A be a ring. The following are equivalent.*

(i) *A is semiperfect.*
(ii) *A is semilocal and each idempotent of $A/\operatorname{rad} A$ can be lifted modulo* $\operatorname{rad} A$.
(iii) *The identity $1 \in A$ is the sum of local orthogonal idempotents.*

Proof. See [Bas60, Theorem 2.1] and [Mül70, Theorem 1]. □

Note that [Bas60] and [Mül70] studied this in the context of non-commutative ring theory. By the above theorem, we get the following proposition in commutative ring theory.

Proposition 3.19 ([Lam01, Theorem 23.11])**.** *Let A be a ring. A is semiperfect if and only if A is a finite direct sum of local rings.*

Proof. Suppose A is semiperfect. By Theorem 3.18, there are orthogonal idempotents $e_1, \dots, e_r$ such that $1 = e_1 + \cdots + e_r$. Thus

$$A \to Ae_1 \oplus \cdots \oplus Ae_r; a \mapsto (ae_1, \dots, ae_r)$$

is an isomorphism. We show the opposite. Let $(A_i, \mathfrak{m}_i)$ be local rings and $A = A_1 \oplus \cdots \oplus A_r$. Suppose $(a_i) + \operatorname{rad} A$ is an idempotent. If $a_i \notin \mathfrak{m}_i$, then $1 - a_i \in \mathfrak{m}_i$ since $a_i - a_i^2 \in \mathfrak{m}_i$. We define b_i by the following equation

$$b_i := \begin{cases} 0 & \text{if } a_i \in \mathfrak{m}_i, \\ 1 & \text{if } a_i \notin \mathfrak{m}_i. \end{cases}$$

Thus $(a_i) + \operatorname{rad} A = (b_i) + \operatorname{rad} A$ and (b_i) is an idempotent in A. □

3.4. **Chase's theorem.**

By Theorem 3.17, an Artinian ring is perfect. In this subsection, we discuss Chase's theorem (Theorem 3.2) for projective modules on Artinian rings.

Let $\mathscr{A}$ be an Abelian category. In $\mathscr{A}$, a direct sum of a family of projective objects is again projective and a direct product of a family of injective objects is again injective. It is natural to ask what happens if we interchange direct sum and direct product. In commutative ring theory, the following theorem is well-known. This fact is mentioned in [BH97, Remark 3.1.4.] without proof. Here we give a proof.

Theorem 3.20. *Let A be a ring. A is a Noetherian ring if and only if for each family of injective modules $\{E_\lambda\}_{\lambda \in \Lambda}$, the direct sum $\bigoplus_{\lambda \in \Lambda} E_\lambda$ is again injective.*

Proof. Suppose A is Noetherian. Let I be an ideal of A and $\varphi : I \to \bigoplus E_\lambda$ a linear map. There is a finite subset $\Lambda' = \{\lambda_1, \dots, \lambda_n\} \subset \Lambda$ such that $\varphi(I) \subset \bigoplus_{\lambda \in \Lambda'} E_\lambda$ since I is finitely generated. Let φ_i the composite of φ and the natural map $\bigoplus E_\lambda \to E_{\lambda_i}$. Now there are $\widetilde{\varphi}_i : A \to E_{\lambda_i}$ which is an extension of φ_i, then

$$\widetilde{\varphi} : A \to \bigoplus_{\lambda \in \Lambda} E_\lambda ; 1 \mapsto \sum_{i=1}^{n} \widetilde{\varphi}_i(1)$$

is an extension of φ. So $\bigoplus E_\lambda$ is injective by the Baer's criterion ([Mat86, Theorem B3]).

We show the opposite. We consider the ascending chain of ideals $I_1 \subset \cdots \subset I_i \subset \cdots$. Let $I = \bigcup I_i$. For each i, there exists an injective module E_i containing A/I_i since $\operatorname{Mod} A$ has enough injectives. The following linear map

$$\varphi : I \to \bigoplus E_i ; a \mapsto (a + I_i)$$

can be extended to $\widetilde{\varphi} : A \to \bigoplus E_i$ since $\bigoplus E_i$ is injective. Let $(x_i) := \widetilde{\varphi}(1)$. The number of i for which $x_i \neq 0$ holds is finite, and let n be its maximum. Then for each $i > n$, $a + I_i = 0$ for any $a \in I$. So A is Noetherian. □

As for a product of projective modules, there are examples over Noetherian rings that is not projective.

Example 3.21. $\prod_{n \in \mathbb{N}} \mathbb{Z}$ is not a projective $\mathbb{Z}$-module. Since any projective module over a PID is free ([HS97, Corollary 5.2]), we need to show that $\prod_{n \in \mathbb{N}} \mathbb{Z}$ is not free. This fact is proved in [Bae37]. Also, a short proof of this fact can be found in [Sch08].

In non-commutative ring theory, Chase ([Cha60]) proved Theorem 3.2 which is the dual of Theorem 3.20. We give a simple proof of a part of Chase's theorem, under the commutativeity condition.

Proposition 3.22. *Let A be an Artinian ring. For each family of projective modules $\{P_\lambda\}_{\lambda\in\Lambda}$, the product $\prod_{\lambda\in\Lambda} P_\lambda$ is again projective.*

Proof. By the structure theorem for Artinian rings ([AM69, Theorem 8.7]), A is a finite direct sum of Artinian local rings. Note that for a product ring $A = \prod A_i$ and $P = \prod P_i \in \operatorname{Mod} A$, P is projective if and only if all P_i are projective. Therefore, we can assume that A is local. Let $\mathfrak{m}$ be the maximal ideal of A, $P := \prod P_\lambda$. We take a composition series of $\mathfrak{m}$

$$0 = \mathfrak{m}_n \subset \cdots \subset \mathfrak{m}_0 = \mathfrak{m}.$$

Note that P is projective if and only if P is free, by the Kaplansky's theorem ([Mat86, Theorem 2.5]). To prove the proposition, it suffices to show the following claim.

For each $0 \le i \le n$, we show that if $P/\mathfrak{m}_i P$ is a free $A/\mathfrak{m}_i$-module then $P/\mathfrak{m}_{i+1}P$ is a free $A/\mathfrak{m}_{i+1}$-module and a basis of $P/\mathfrak{m}_i P$ can be lifted to $P/\mathfrak{m}_{i+1}P$.

We prove this by induction. Let $\{e_{\lambda'}\}_{\lambda'\in\Lambda'}$ be a basis of the $A/\mathfrak{m}$-linear space $P/\mathfrak{m}P$. Suppose $P/\mathfrak{m}_i P \cong \bigoplus_{\lambda'\in\Lambda'}(A/\mathfrak{m}_i)e_{\lambda'}$. For each $a \in \mathfrak{m}_i \setminus \mathfrak{m}_{i+1}$, $a\cdot : P/\mathfrak{m}P \to \mathfrak{m}_i P/\mathfrak{m}_{i+1}P$ is an isomorphism. The reason is that the following equations

$$P/\mathfrak{m}P = \prod P_\lambda/\mathfrak{m}P_\lambda, \quad \mathfrak{m}_i P/\mathfrak{m}_{i+1}P = \prod \mathfrak{m}_i P_\lambda/\mathfrak{m}_{i+1}P_\lambda$$

hold and we can assume that $P_\lambda = A$ since P_λ is free. So

$$\bigoplus(\mathfrak{m}_i/\mathfrak{m}_{i+1})e_{\lambda'} \cong \bigoplus(A/\mathfrak{m})e_{\lambda'} \cong \mathfrak{m}_i P/\mathfrak{m}_{i+1}P.$$

Thus, the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus(\mathfrak{m}_i/\mathfrak{m}_{i+1})e_{\lambda'} & \longrightarrow & \bigoplus(A/\mathfrak{m}_{i+1})e_{\lambda'} & \longrightarrow & \bigoplus(A/\mathfrak{m}_i)e_{\lambda'} & \longrightarrow & 0 \\
 & & \downarrow\wr\!\!\!\; & & \downarrow & & \downarrow\wr\!\!\!\; & & \\
0 & \longrightarrow & \mathfrak{m}_i P/\mathfrak{m}_{i+1}P & \longrightarrow & P/\mathfrak{m}_{i+1}P & \longrightarrow & P/\mathfrak{m}_i P & \longrightarrow & 0
\end{array}$$

is commutative and the five lemma implies $P/\mathfrak{m}_{i+1}P \cong \bigoplus(A/\mathfrak{m}_{i+1})e_{\lambda'}$. Thus, the above claim is proved. ☐

3.5. **Hierarchy of classes of rings around perfectness.**

In proof of Theorem 3.2, Chase proved that A satisfies DCCP (Descending Chain Condition of Principal ideals) if a direct product of the projective modules is again projective [Cha60, Theorem 3.1]. By [Bas60, Theorem P](Theorem 3.17), a ring A is perfect if and only if A satisfies DCCP. So Chase's theorem can also be considered as an argument related to perfect rings. In this subsection, we discuss the hierarchy of classes of rings around perfectness.

Proposition 3.23. *Let A be a ring. If A satisfies DCCP, then* $\dim A = 0$.

Proof. We take each $P \in \operatorname{Spec} A$. It suffices to show that A/P is a field. For each $a \notin P$, there is an $n > 0$ such that $(a^n) = (a^{n+1})$. So there exists an element $b \in A$ such that $a^{n+1}b - a^n = a^n(ab-1) = 0 \in P$. Since $a \notin P, ab - 1 \in P$. Therefore $ab + P = 1 + P \in A/P$. Thus A/P is a field. ☐

Corollary 3.24. *Let A be a ring. A is Artinian if and only if A is Noetherian and perfect.*

Proof. It is clear that the Artinian ring is perfect by Theorem 3.17. Also if A is perfect then $\dim A = 0$ by the above proposition. So a Noetherian perfect ring is Artinian. ☐

For the class of 0-dimensional Gorenstein rings which is smaller than the class of Artinian rings, we can show the following proposition.

Proposition 3.25. *Let A be a ring. We denote by* $\operatorname{Prj} A$ *the category of projective A-modules and by* $\operatorname{Inj} A$ *the category of injective A-modules. A is Gorenstein and* $\dim A = 0$ *if and only if* $\operatorname{Prj} A = \operatorname{Inj} A$ *as subcategories of* $\operatorname{Mod} A$.

Proof. Suppose A is Gorenstein. First, we show that all projective modules are injective. It suffices to show that $\operatorname{Hom}_A(A, A) \to \operatorname{Hom}_A(I, A)$ is surjective for any ideal I by Baer's Criterion. Since surjectivity is a local property and [Mat86, Themrem 7.11], it suffices to consider the localisation for any $P \in \operatorname{Spec} A$. Now $\operatorname{Hom}_{A_P}(A_P, A_P) \to \operatorname{Hom}_{A_P}(I_P, A_P)$ is surjective since A_P is injective. So A is injective. Therefore, all free modules are injective, and projective modules are their direct summand, so they are injective. Next, we show that all injective modules are projective. By structure theorem for injective modules over Noetherian rings ([BH97, Theorem 3.2.8]), every injective A-module can be written in the form of a direct sum of $E(A/P)$ $(P \in \operatorname{Spec} A)$. Furthermore;

$$A = \bigoplus_{P \in \operatorname{Spec} A} E(A/P)^{\mu^0(P,A)}, \quad \mu^0(P, A) := \dim_{k(P)} \operatorname{Hom}_{A_P}(k(P), A_P) = 1.$$

So for each $P \in \operatorname{Spec} A$, $E(A/P)$ is a direct summand of A. Thus $E(A/P)$ is projective.

We show the opposite. By Theorem 3.20, A is Noetherian. Also for each $P \in \operatorname{Spec} A$, A_P is injective by Lemma 1.14. So A is 0-dimensional Gorenstein. □

In summary, there is the following relationship between the several classes of rings.

$$\begin{array}{ccc}
\text{Semi-perfect} & \overset{\text{Proposition 3.22}}{\Longleftrightarrow} & \text{Finite sum of local rings} \\
\Uparrow & & \\
\text{Perfect} & \overset{\text{Theorem 3.17}}{\Longleftrightarrow} & \text{DCCP} \\
\Uparrow & & \\
\text{Artinian} & \overset{\text{Corollary 3.24}}{\Longleftrightarrow} & \text{Noetherian and Perfect} \\
\Uparrow & & \\
\text{0-dimensional Gorenstein} & \overset{\text{Proposition 3.25}}{\Longleftrightarrow} & \text{Prj} = \text{Inj}
\end{array}$$

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